Refined F_5 algorithms for ideals of minors of square matrices

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Determinantal systems

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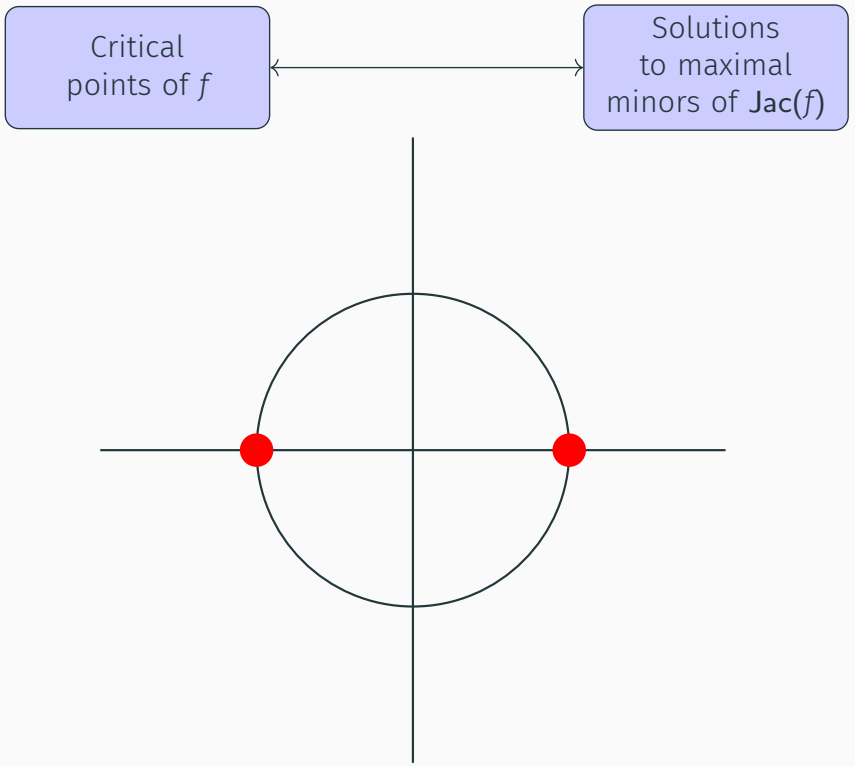

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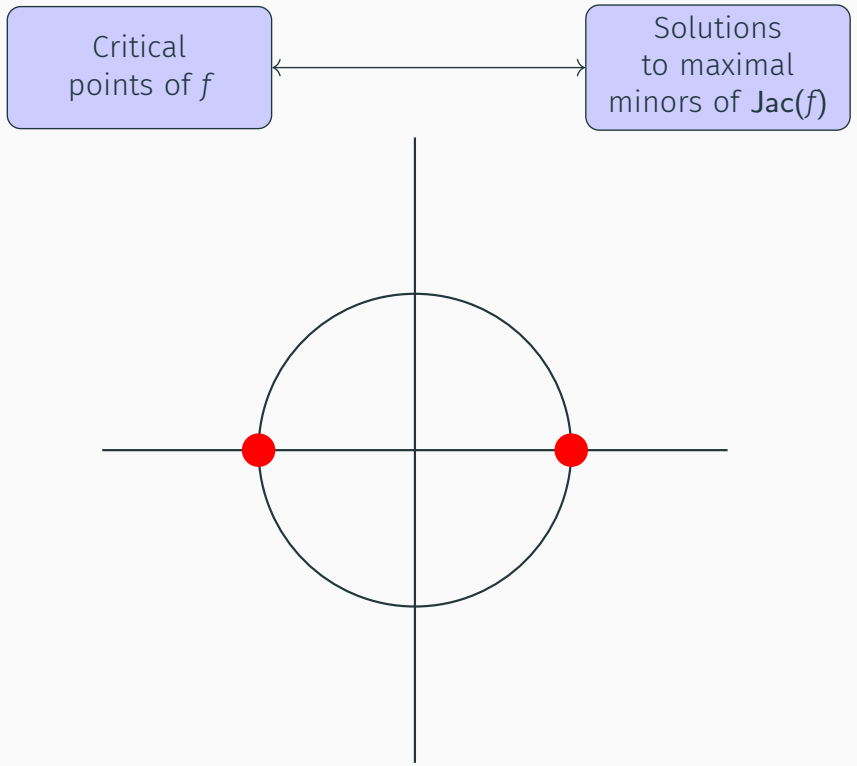
Post-quantum crypto
(multivariate and code-based cryptography)

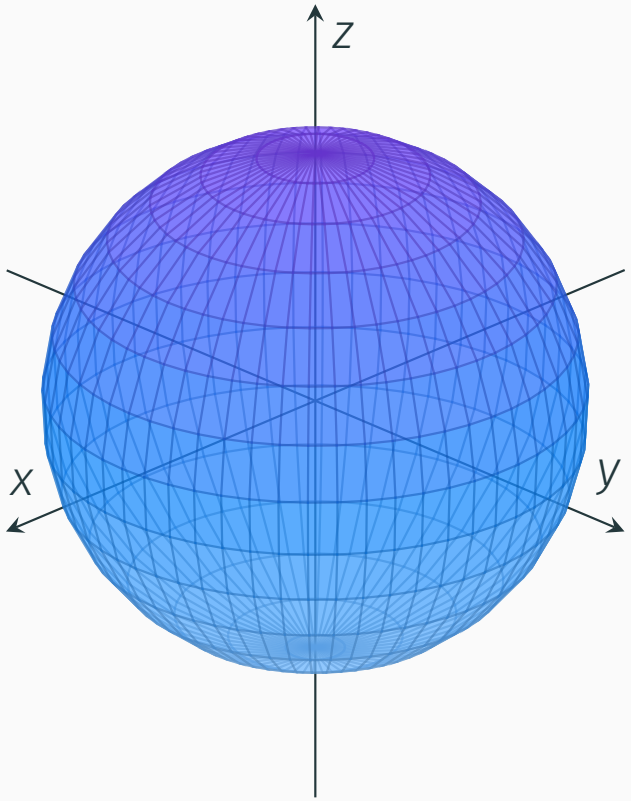


The MinRank Problem

$f_{i,j}$ are **linear forms** in $\mathbb{k}[x_1, \ldots, x_k]$.
Find $a \in \overline{\mathbb{k}}^k$ with $\mathbf{rank}(M(a)) \leq r$.

Polynomial systems solving

$$f_1 = x^2 + y^2 + z^2 - 1$$



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$$f_1 = x^2 + y^2 + z^2 - 1$$
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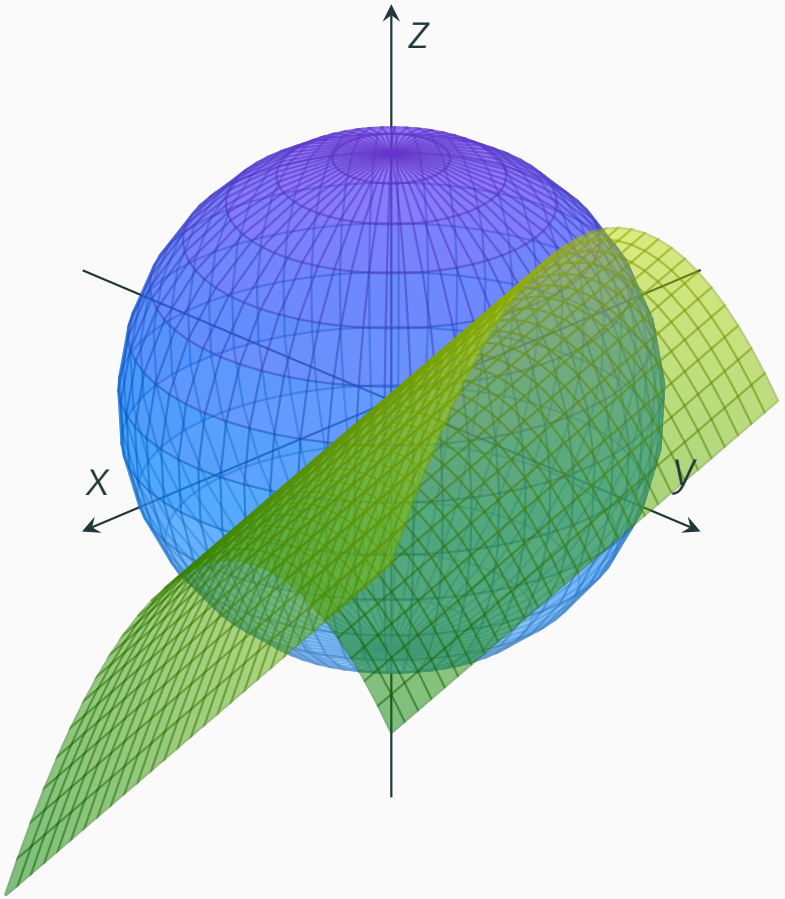

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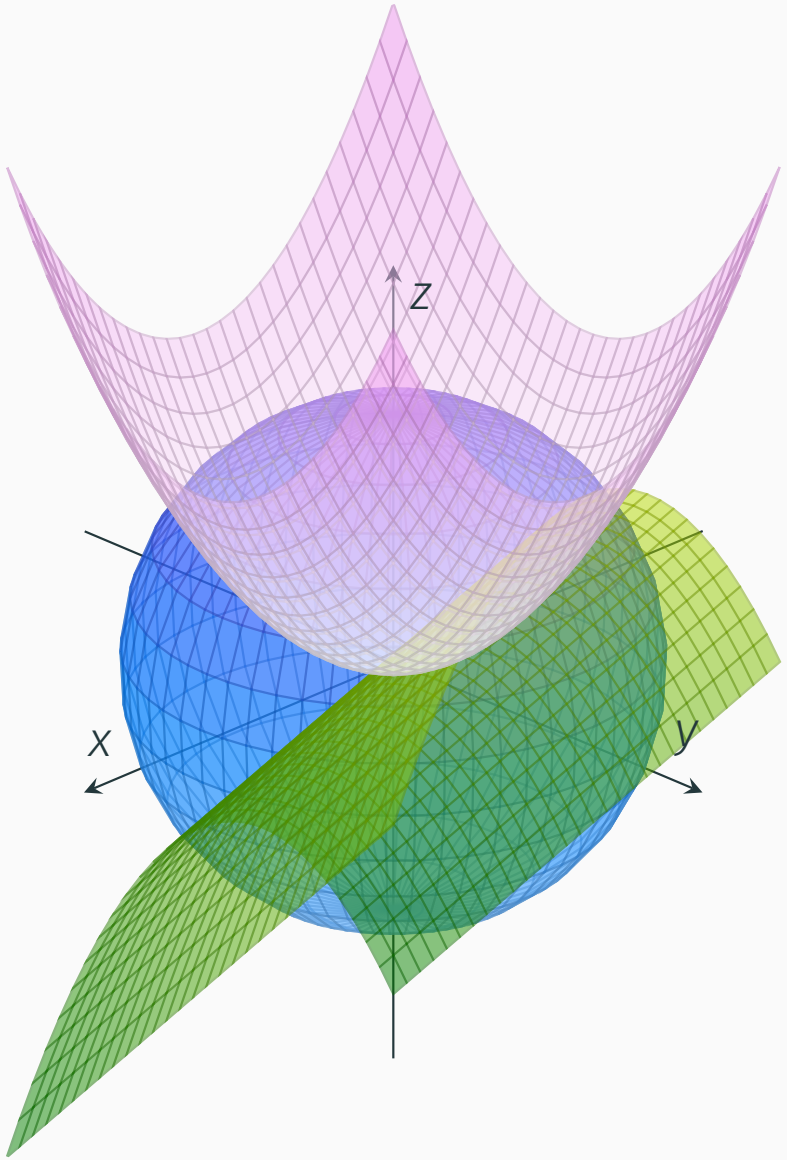

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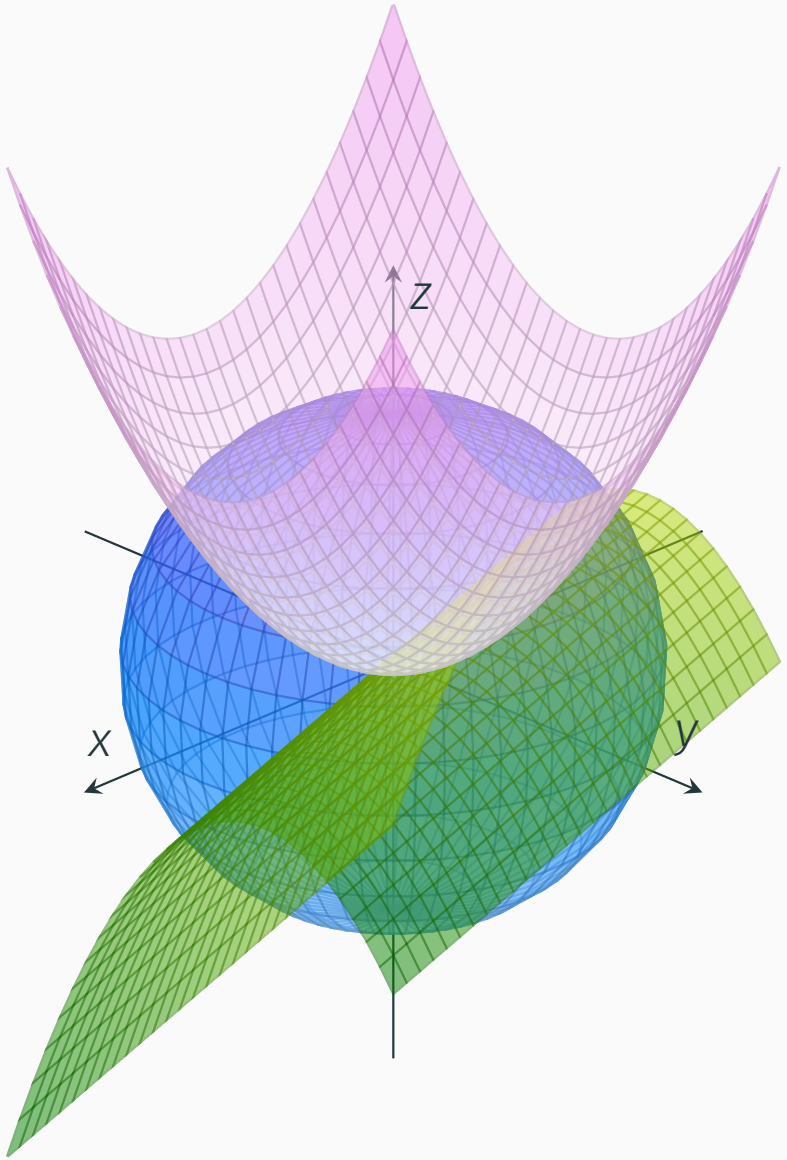
Gröbner basis algorithm

$$x^2 - y + z$$
$$y^2 + y - 2z$$
$$z^2 + z - 1$$



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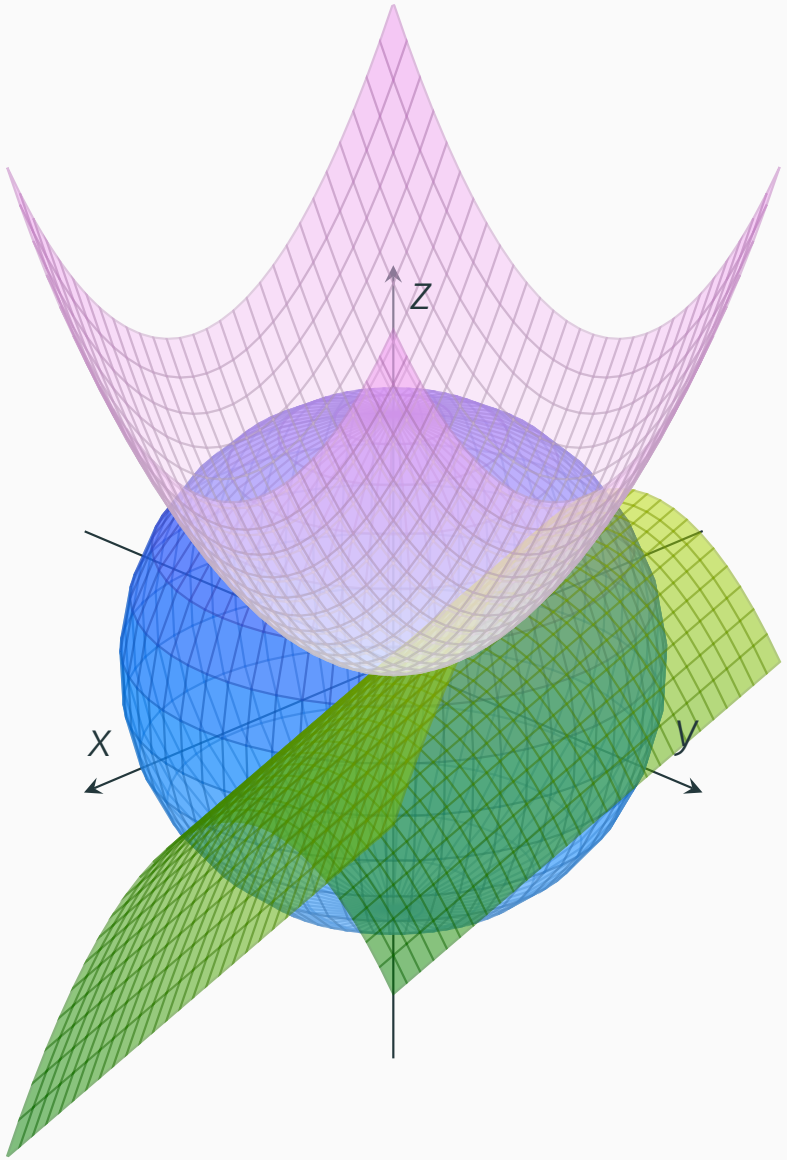

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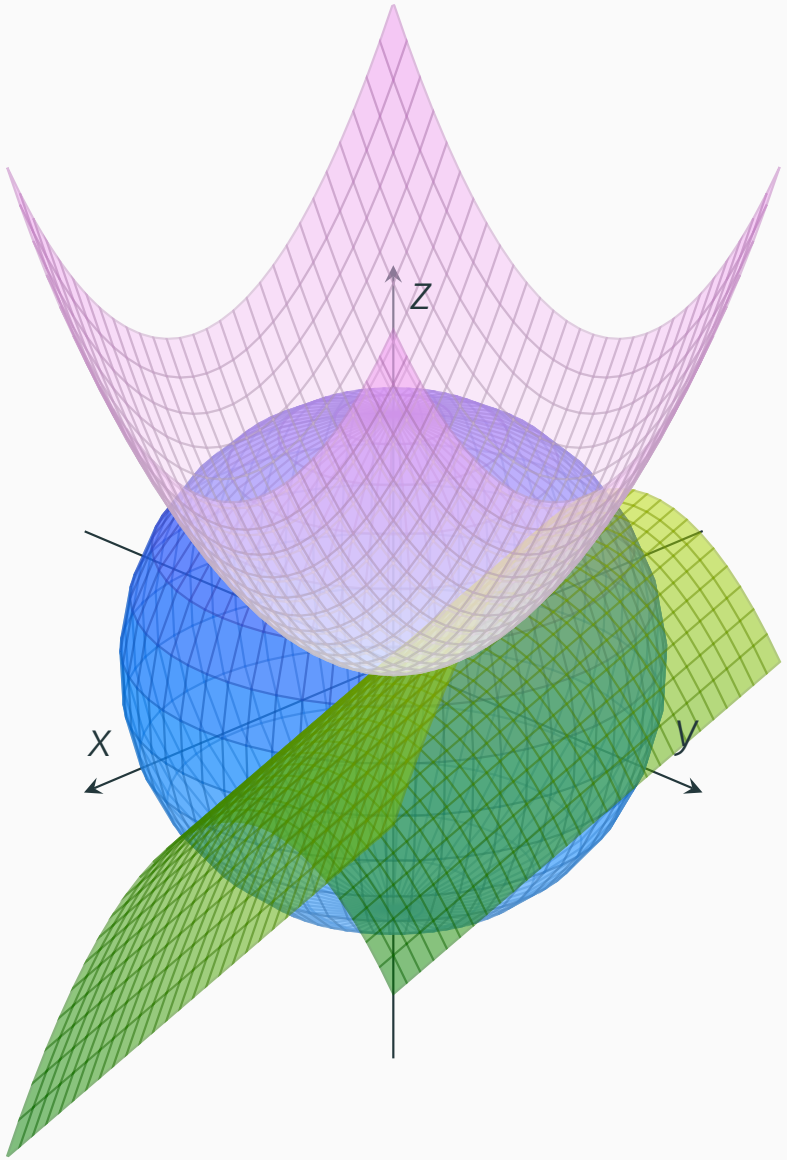

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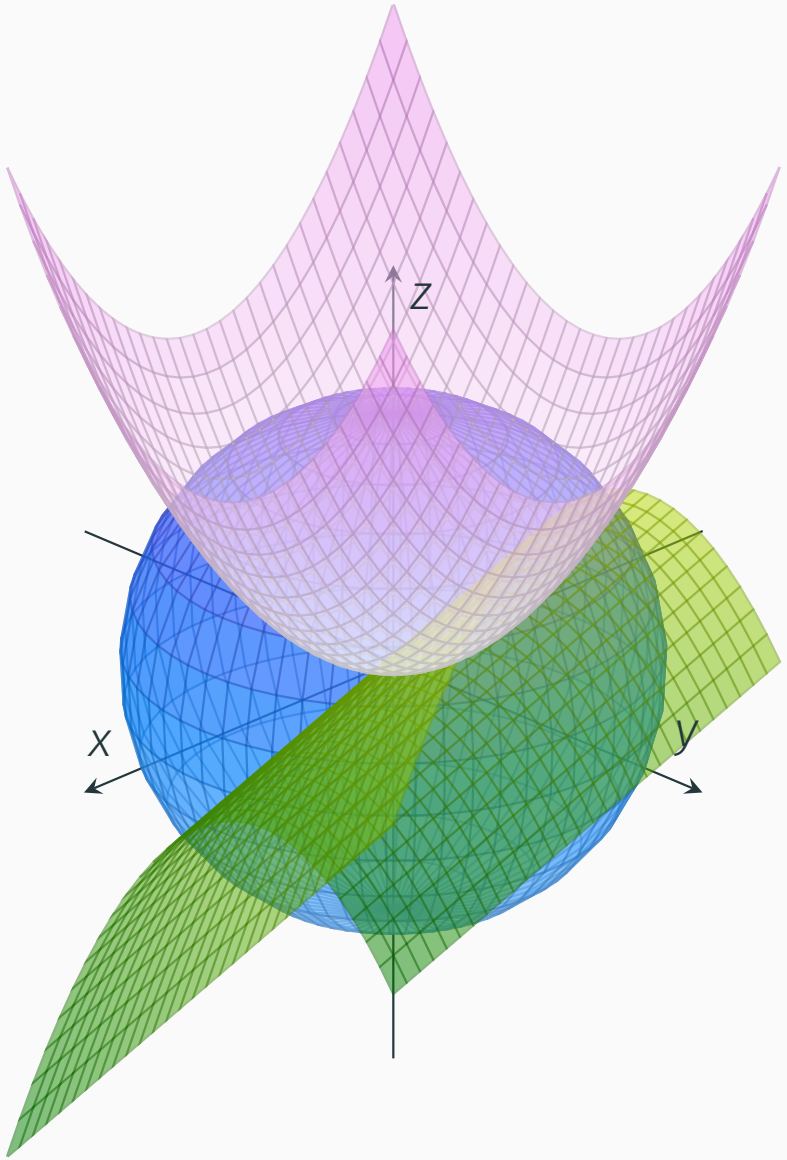

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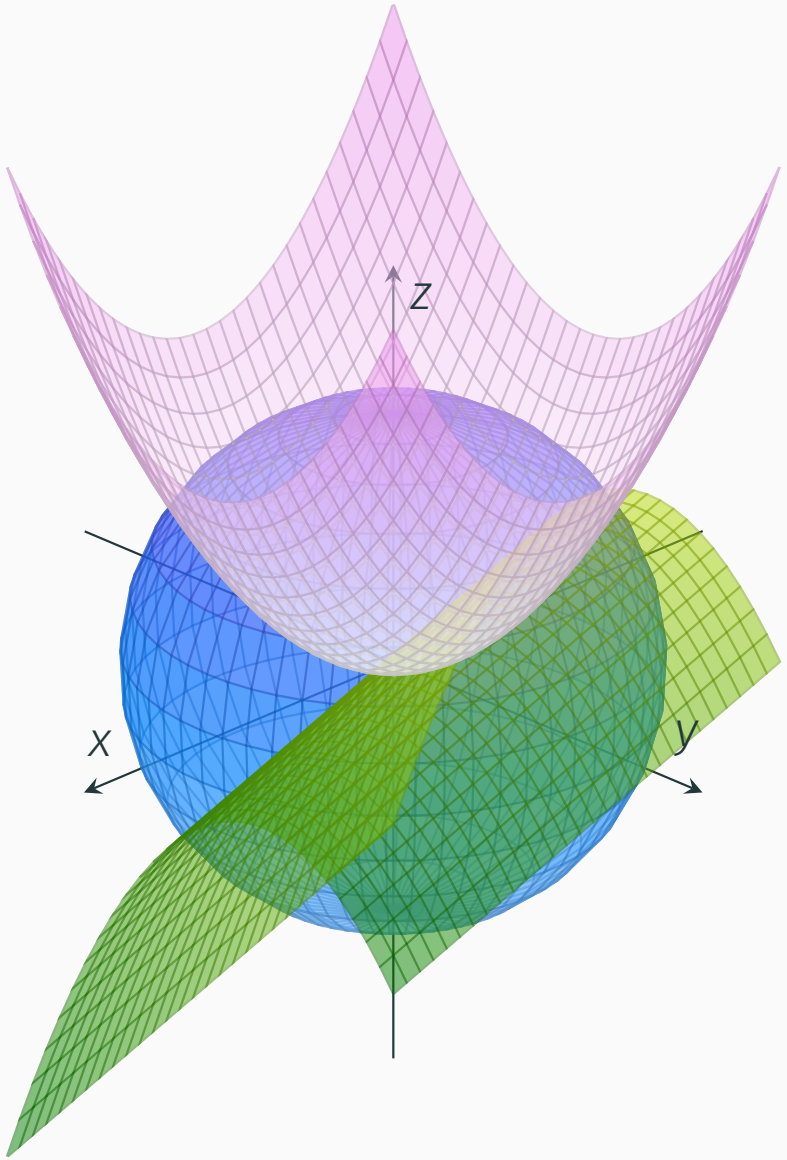

Macaulay matrices - linearization

Assume F homogeneous.

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$$\begin{cases} f_1 = 2x^2 + 11xy - y^2 \\ f_2 = 4x^2 + xy - 2y^2 \\ f_3 = -6x^2 - xy + y^2 \end{cases}$$

$$\begin{array}{c} \begin{array}{cccc} x^3 & x^2y & xy^2 & y^3 \end{array} \\ \begin{pmatrix} & & & \\ & & & \\ & & & \end{pmatrix} \end{array}$$

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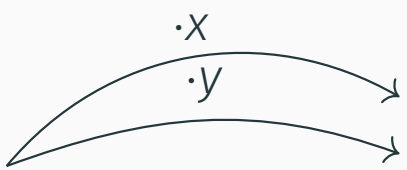

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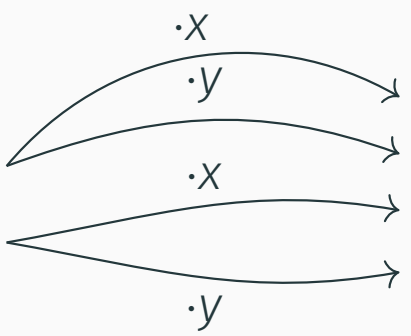

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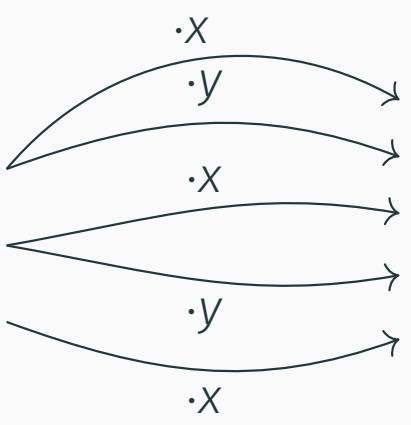

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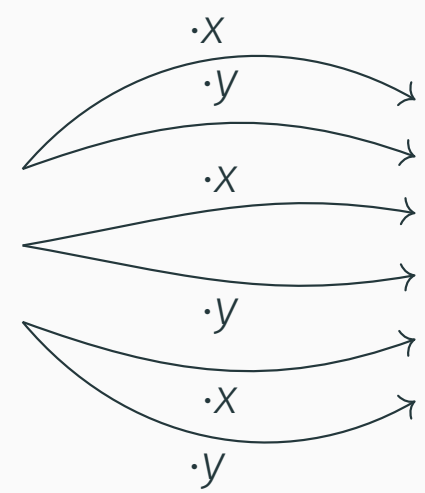

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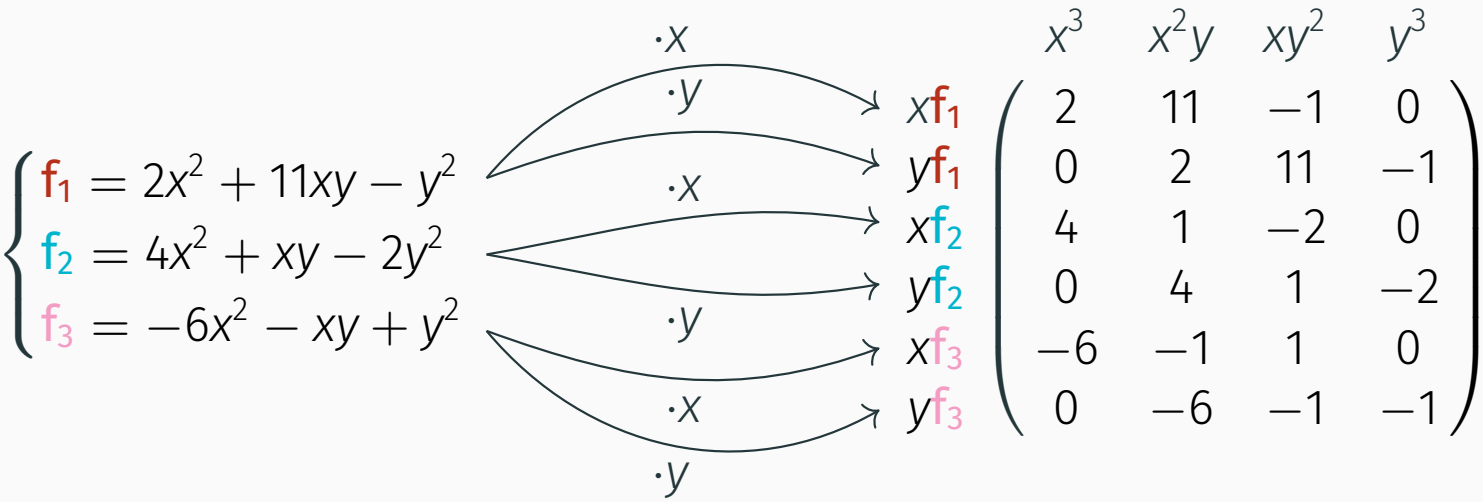

Theorem (Macaulay bound, [Lazard, 1983])

The maximum degree of a polynomial in the grevlex Gröbner basis of a generic polynomial system $f_1, \ldots, f_m$ is

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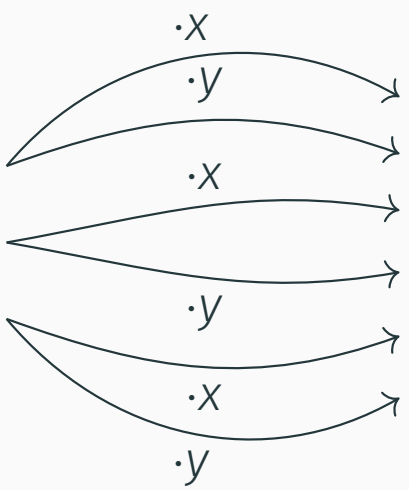

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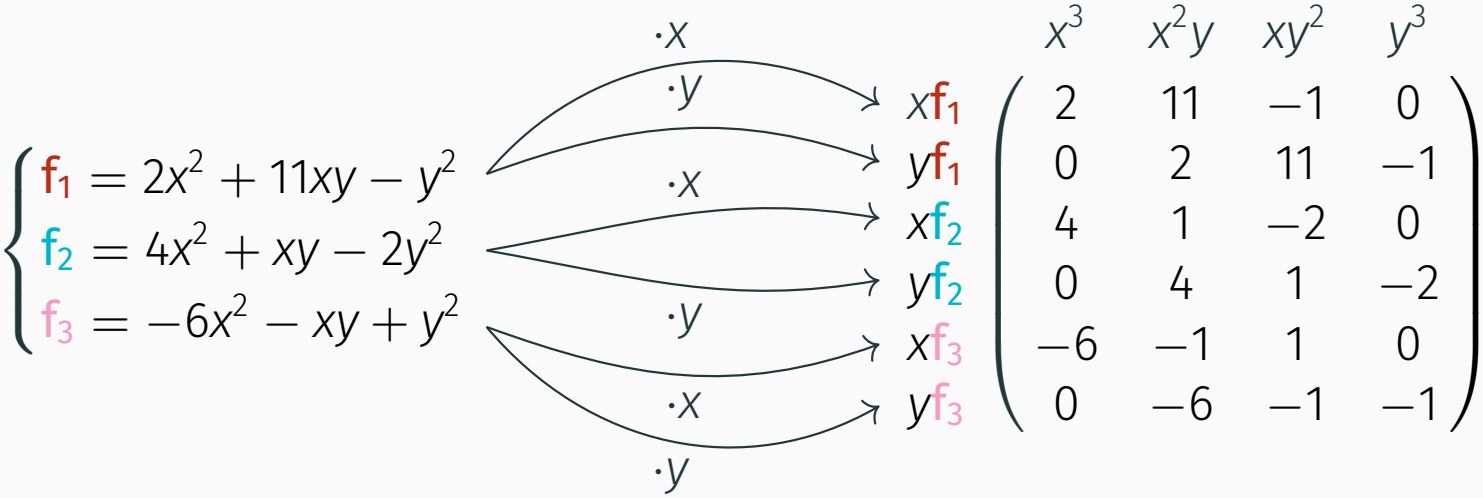

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The rows of the echelonization of the Macaulay matrix of F in degree d form the elements of degree d of a $\succ$-Gröbner basis for F.

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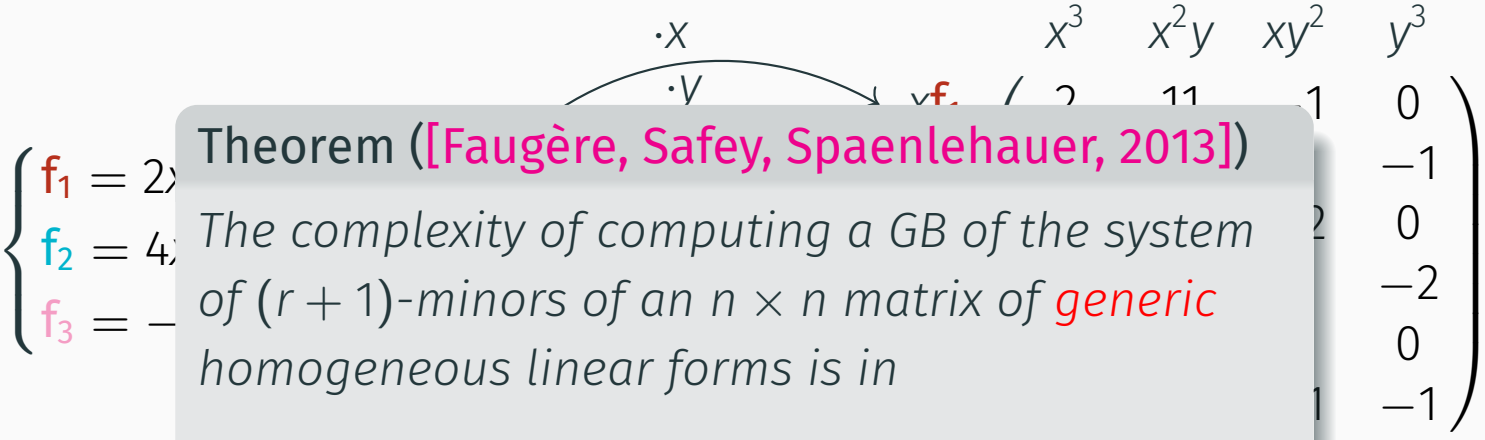

Theorem ([Faugère, Safey, Spaenlehauer, 2013])

The complexity of computing a GB of the system of $(r+1)$-minors of an $n \times n$ matrix of generic homogeneous linear forms is in

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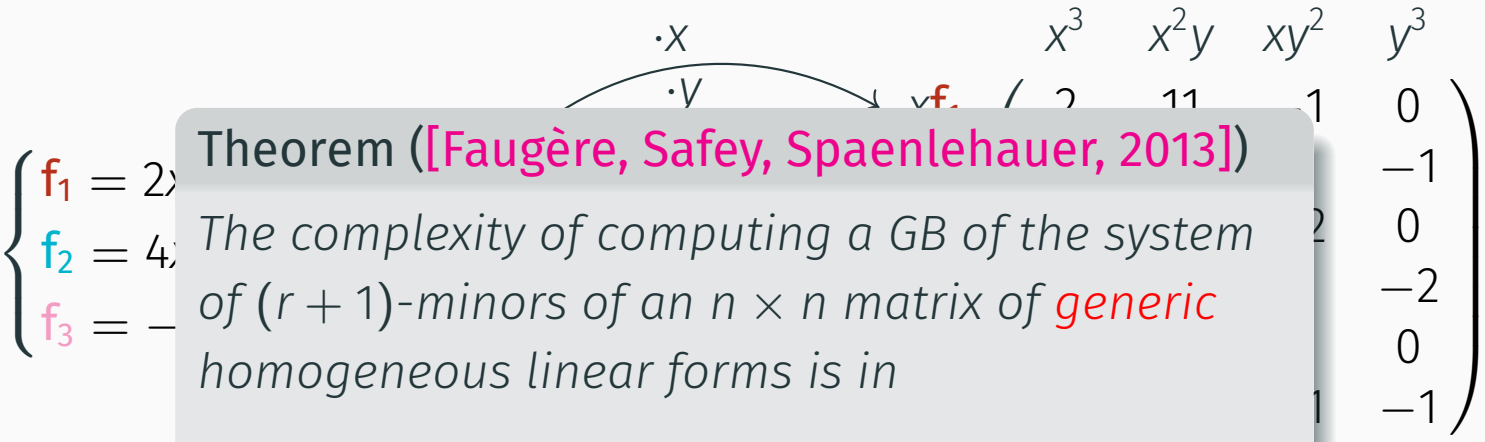

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This upper bound is not very sharp since the Macaulay matrices are not full rank!

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The F_5 algorithm ([Faugère, 2002])

Let $\mathbb{k} = \mathbb{F}_7$, $\succ =$ grevlex .

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$$\widetilde{\mathcal{M}_2}$$

$$\begin{array}{c} \\ (1,1) \\ (2,1) \\ (3,1) \end{array} \begin{array}{c} \begin{array}{cccccc} x^2 & xy & y^2 & xz & yz & z^2 \end{array} \\ \begin{pmatrix} 1 & 1 & 2 & 1 & 1 & 4 \\ 0 & 1 & 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 2 & 0 & 4 \end{pmatrix} \end{array}$$

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$\widetilde{\mathcal{M}_2}$

	x^2	xy	y^2	xz	yz	z^2
$(1,1)$	1	1	2	1	1	4
$(2,1)$	0	1	0	0	2	4
$(3,1)$	0	0	1	2	0	4

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
$(1,x^2)$	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
$(1,xy)$	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
$(1,y^2)$	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
$(1,xz)$	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
$(1,yz)$	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
$(1,z^2)$	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
$(2,x^2)$	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
$(2,xy)$	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
$(2,y^2)$	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
$(2,yz)$	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
$(2,z^2)$	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
$(3,x^2)$	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
$(3,xy)$	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
$(3,y^2)$	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
$(3,xz)$	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
$(3,yz)$	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
$(3,z^2)$	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

The F_5 algorithm ([Faugère, 2002])

Let $\mathbb{k} = \mathbb{F}_7$, $\succ =$ grevlex .

$$f_1 = 5x^2 + 5xy + 3y^2 + 5xz + 5yz + 6z^2$$
$$f_2 = 2x^2 + xy + 4y^2 + 2xz + 4z^2$$
$$f_3 = 4x^2 + xy + 4y^2 + 3xz + 5yz + 2z^2$$

$\widetilde{\mathcal{M}_2}$

	x^2	xy	y^2	xz	yz	z^2
$(1,1)$	1	1	2	1	1	4
$(2,1)$	0	1	0	0	2	4
$(3,1)$	0	0	1	2	0	4

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
$(1,x^2)$	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
$(1,xy)$	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
$(1,y^2)$	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
$(1,xz)$	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
$(1,yz)$	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
$(1,z^2)$	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
$(2,x^2)$	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
$(2,xy)$	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
$(2,y^2)$	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
$(2,yz)$	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
$(2,z^2)$	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
$(3,x^2)$	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
$(3,xy)$	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
$(3,y^2)$	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
$(3,xz)$	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
$(3,yz)$	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
$(3,z^2)$	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

The F_5 algorithm ([Faugère, 2002])

Let $\mathbb{k} = \mathbb{F}_7$, $\succ =$ grevlex .

$$f_1 = 5x^2 + 5xy + 3y^2 + 5xz + 5yz + 6z^2$$
$$f_2 = 2x^2 + xy + 4y^2 + 2xz + 4z^2$$
$$f_3 = 4x^2 + xy + 4y^2 + 3xz + 5yz + 2z^2$$

$\widetilde{\mathcal{M}_2}$

	x^2	xy	y^2	xz	yz	z^2
$(1,1)$	1	1	2	1	1	4
$(2,1)$	0	1	0	0	2	4
$(3,1)$	0	0	1	2	0	4

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
$(1,x^2)$	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
$(1,xy)$	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
$(1,y^2)$	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
$(1,xz)$	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
$(1,yz)$	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
$(1,z^2)$	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
$(2,x^2)$	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
$(2,xy)$	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
$(2,y^2)$	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
$(2,yz)$	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
$(2,z^2)$	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
$(3,x^2)$	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
$(3,xy)$	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
$(3,y^2)$	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
$(3,xz)$	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
$(3,yz)$	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
$(3,z^2)$	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

The F_5 algorithm ([Faugère, 2002])

Let $\mathbb{k} = \mathbb{F}_7$, $\succ =$ grevlex .

$$f_1 = 5x^2 + 5xy + 3y^2 + 5xz + 5yz + 6z^2$$
$$f_2 = 2x^2 + xy + 4y^2 + 2xz + 4z^2$$
$$f_3 = 4x^2 + xy + 4y^2 + 3xz + 5yz + 2z^2$$

$\widetilde{\mathcal{M}_2}$

	x^2	xy	y^2	xz	yz	z^2
$(1,1)$	1	1	2	1	1	4
$(2,1)$	0	1	0	0	2	4
$(3,1)$	0	0	1	2	0	4

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
$(1,x^2)$	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
$(1,xy)$	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
$(1,y^2)$	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
$(1,xz)$	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
$(1,yz)$	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
$(1,z^2)$	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
$(2,x^2)$	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
$(2,xy)$	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
$(2,y^2)$	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
$(2,yz)$	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
$(2,z^2)$	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
$(3,x^2)$	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
$(3,xy)$	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
$(3,y^2)$	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
$(3,xz)$	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
$(3,yz)$	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
$(3,z^2)$	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

The F_5 algorithm ([Faugère, 2002])

Let $\mathbb{k} = \mathbb{F}_7$, $\succ =$ grevlex .

$$f_1 = 5x^2 + 5xy + 3y^2 + 5xz + 5yz + 6z^2$$
$$f_2 = 2x^2 + xy + 4y^2 + 2xz + 4z^2$$
$$f_3 = 4x^2 + xy + 4y^2 + 3xz + 5yz + 2z^2$$

$\widetilde{\mathcal{M}_2}$

	x^2	xy	y^2	xz	yz	z^2
$(1,1)$	1	1	2	1	1	4
$(2,1)$	0	1	0	0	2	4
$(3,1)$	0	0	1	2	0	4

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
$(1,x^2)$	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
$(1,xy)$	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
$(1,y^2)$	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
$(1,xz)$	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
$(1,yz)$	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
$(1,z^2)$	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
$(2,x^2)$	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
$(2,xy)$	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
$(2,y^2)$	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
$(2,yz)$	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
$(2,z^2)$	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
$(3,x^2)$	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
$(3,xy)$	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
$(3,y^2)$	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
$(3,xz)$	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
$(3,yz)$	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
$(3,z^2)$	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

The F_5 algorithm ([Faugère, 2002])

Let $\mathbb{k} = \mathbb{F}_7$, $\succ =$ grevlex .

$$f_1 = 5x^2 + 5xy + 3y^2 + 5xz + 5yz + 6z^2$$
$$f_2 = 2x^2 + xy + 4y^2 + 2xz + 4z^2$$
$$f_3 = 4x^2 + xy + 4y^2 + 3xz + 5yz + 2z^2$$

$\widetilde{\mathcal{M}_2}$

	x^2	xy	y^2	xz	yz	z^2
$(1,1)$	1	1	2	1	1	4
$(2,1)$	0	1	0	0	2	4
$(3,1)$	0	0	1	2	0	4

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
$(1,x^2)$	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
$(1,xy)$	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
$(1,y^2)$	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
$(1,xz)$	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
$(1,yz)$	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
$(1,z^2)$	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
$(2,x^2)$	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
$(2,xy)$	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
$(2,y^2)$	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
$(2,yz)$	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
$(2,z^2)$	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
$(3,x^2)$	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
$(3,xy)$	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
$(3,y^2)$	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
$(3,xz)$	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
$(3,yz)$	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
$(3,z^2)$	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

The F_5 algorithm ([Faugère, 2002])

Let $\mathbb{k} = \mathbb{F}_7$, $\succ =$ grevlex .

$$f_1 = 5x^2 + 5xy + 3y^2 + 5xz + 5yz + 6z^2$$
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$$f_3 = 4x^2 + xy + 4y^2 + 3xz + 5yz + 2z^2$$

$\widetilde{\mathcal{M}_2}$

	x^2	xy	y^2	xz	yz	z^2
$(1,1)$	1	1	2	1	1	4
$(2,1)$	0	1	0	0	2	4
$(3,1)$	0	0	1	2	0	4

Lazard: $\mathcal{M}_4$ is 18×15.

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
$(1,x^2)$	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
$(1,xy)$	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
$(1,y^2)$	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
$(1,xz)$	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
$(1,yz)$	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
$(1,z^2)$	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
$(2,x^2)$	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
$(2,xy)$	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
$(2,y^2)$	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
$(2,yz)$	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
$(2,z^2)$	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
$(3,x^2)$	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
$(3,xy)$	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
$(3,y^2)$	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
$(3,xz)$	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
$(3,yz)$	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
$(3,z^2)$	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

The F_5 algorithm ([Faugère, 2002])

Let $\mathbb{k} = \mathbb{F}_7$, $\succ =$ grevlex .

$$f_1 = 5x^2 + 5xy + 3y^2 + 5xz + 5yz + 6z^2$$
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$$f_3 = 4x^2 + xy + 4y^2 + 3xz + 5yz + 2z^2$$

$\widetilde{\mathcal{M}_2}$

	x^2	xy	y^2	xz	yz	z^2
$(1,1)$	1	1	2	1	1	4
$(2,1)$	0	1	0	0	2	4
$(3,1)$	0	0	1	2	0	4

Lazard: $\mathcal{M}_4$ is 18×15.

F5: $\mathcal{M}_4$ is 15×15 and full rank!

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
$(1,x^2)$	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
$(1,xy)$	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
$(1,y^2)$	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
$(1,xz)$	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
$(1,yz)$	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
$(1,z^2)$	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
$(2,x^2)$	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
$(2,xy)$	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
$(2,y^2)$	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
$(2,yz)$	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
$(2,z^2)$	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
$(3,x^2)$	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
$(3,xy)$	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
$(3,y^2)$	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
$(3,xz)$	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
$(3,yz)$	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
$(3,z^2)$	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

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Let $\mathbb{k} = \mathbb{F}_7$, $\succ$ = grevlex.

$$f_1 = 5x^2 + 5xy + 3y^2 + 5xz + 5yz + 6z^2$$
$$f_2 = 2x^2 + xy + 4y^2 + 2xz + 4z^2$$
$$f_3 = 4x^2 + xy + 4y^2 + 3xz + 5yz + 2z^2$$

$(f_1, \ldots, f_m)$ generic

$\widetilde{\mathcal{M}_2}$

	x^2	xy	y^2	xz	yz	z^2
$(1,1)$	1	1	2	1	1	4
$(2,1)$	0	1	0	0	2	4
$(3,1)$	0	0	1	2	0	4

Lazard: $\mathcal{M}_4$ is 18×15.

F5: $\mathcal{M}_4$ is 15×15 and full rank!

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
$(1,x^2)$	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
$(1,xy)$	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
$(1,y^2)$	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
$(1,xz)$	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
$(1,yz)$	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
$(1,z^2)$	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
$(2,x^2)$	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
$(2,xy)$	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
$(2,y^2)$	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
$(2,yz)$	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
$(2,z^2)$	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
$(3,x^2)$	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
$(3,xy)$	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
$(3,y^2)$	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
$(3,xz)$	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
$(3,yz)$	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
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$\Downarrow$

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	x^2	xy	y^2	xz	yz	z^2
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Lazard: $\mathcal{M}_4$ is 18×15.

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	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
$(1,x^2)$	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
$(1,xy)$	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
$(1,y^2)$	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
$(1,xz)$	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
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$(2,y^2)$	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
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$(1,x^2)$	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
$(1,xy)$	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
$(1,y^2)$	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
$(1,xz)$	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
$(1,yz)$	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
$(1,z^2)$	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
$(2,x^2)$	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
$(2,xy)$	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
$(2,y^2)$	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
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$(f_1, \ldots, f_m)$ generic

$\Downarrow$

- **No** reductions to zero.
- Precise complexity analysis [1]

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$(3,y^2)$	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
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How do we remove reductions to zero?

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									5	5	0	6	0	0	
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$(2,x^2)$	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
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$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
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Contributions

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M is an $n \times n$ matrix of generic linear forms over $\mathbb{k}[x_1, \ldots, x_k]$, $r \leq n - 1$. Let $F_r(M)$ be the system of $(r+1)$-minors of M. Suppose $F_r(M)$ is zero-dimensional.

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New F_5-type criteria

- Allows us to avoid all reductions to zero in degree $r+2$

$$\rightsquigarrow \binom{n}{r+2}^2 \left(\frac{2(r+2)(r+1)}{n-r-1} + 2r + 2 \right) \text{ reductions avoided.}$$

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Theorem ([G., Neiger, Safey El Din, 2023])

The complexity of computing a grevlex Gröbner basis for the system of $(n-1)$-minors of M is in

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Homogeneous: $O(n^{4\omega-1}) \rightsquigarrow \begin{matrix} O(n^{2\omega+3}) \\ \Omega(n^6) \end{matrix}$

Affine: $O(n^{4\omega})$

Syzygies

Definition (Syzygy)

$(a_1, \ldots, a_m) \in \Bbbk[x_1, \ldots, x_k]^m$ with

$$a_1 f_1 + \cdots + a_m f_m = 0$$

is called a **syzygy** of $f_1, \ldots, f_m$.

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Example (Koszul syzygies)

$$f_i = \mathrm{LT}(f_i) + \mathrm{tail}(f_i)$$

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Theorem ([Hilbert, 1890])

Free resolution $0 \to \mathcal{E}_\ell \xrightarrow{d_\ell} \mathcal{E}_{\ell-1} \xrightarrow{d_{\ell-1}} \cdots \to \mathcal{E}_1 \xrightarrow{d_1} \mathcal{E}_0 \xrightarrow{\epsilon} \langle F \rangle \to 0 \implies$

$$\mathrm{Syz}_k(F) = \ker(d_k) = \mathrm{im}(d_{k+1}).$$

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m_{ij} = determinant of submatrix of M given by deleting i-th row, j-th column.

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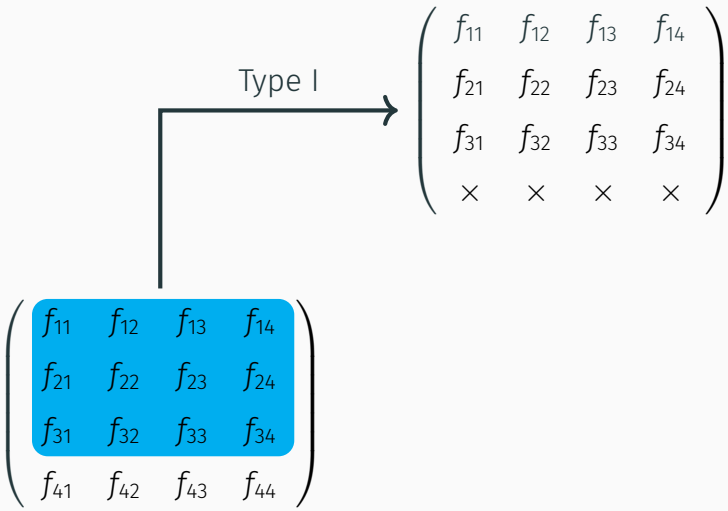

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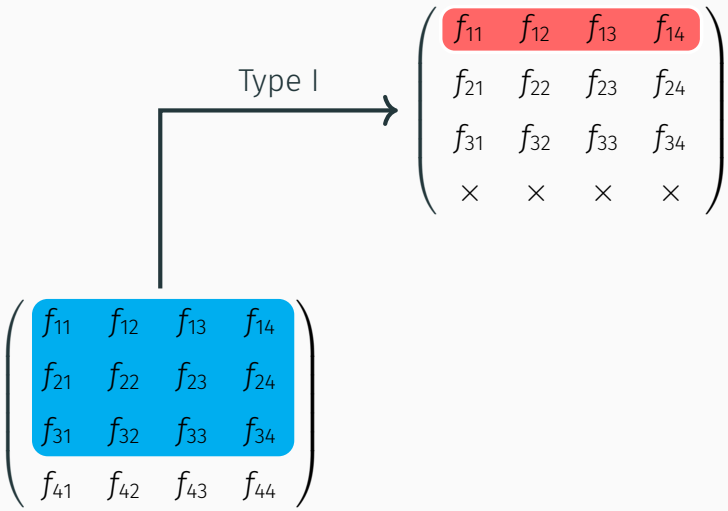

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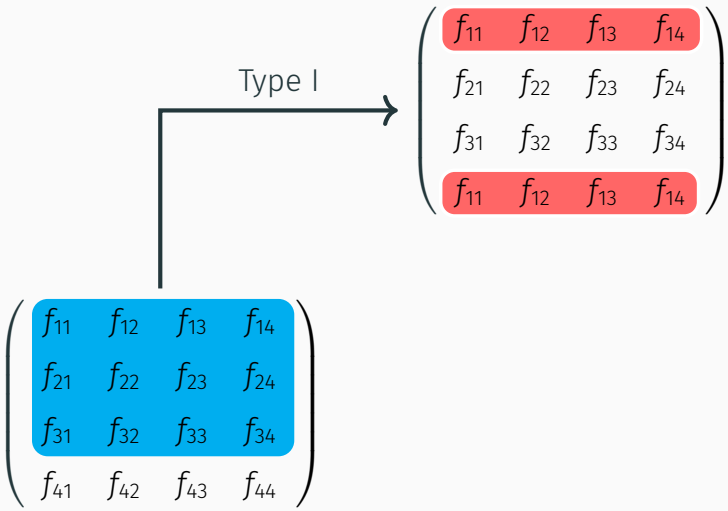

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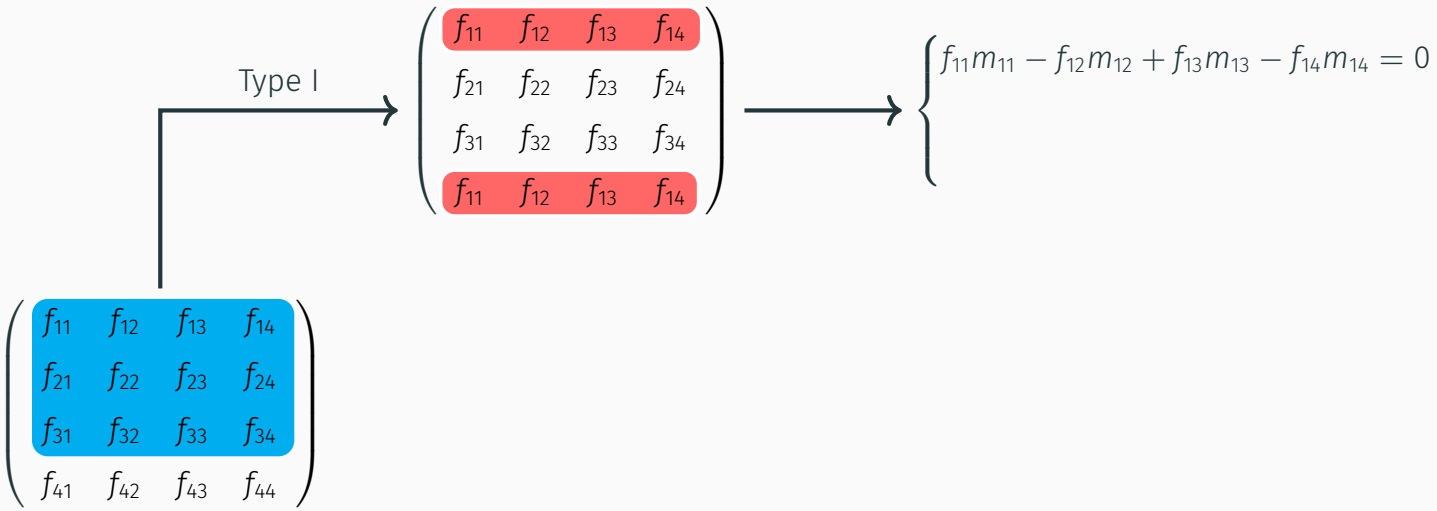

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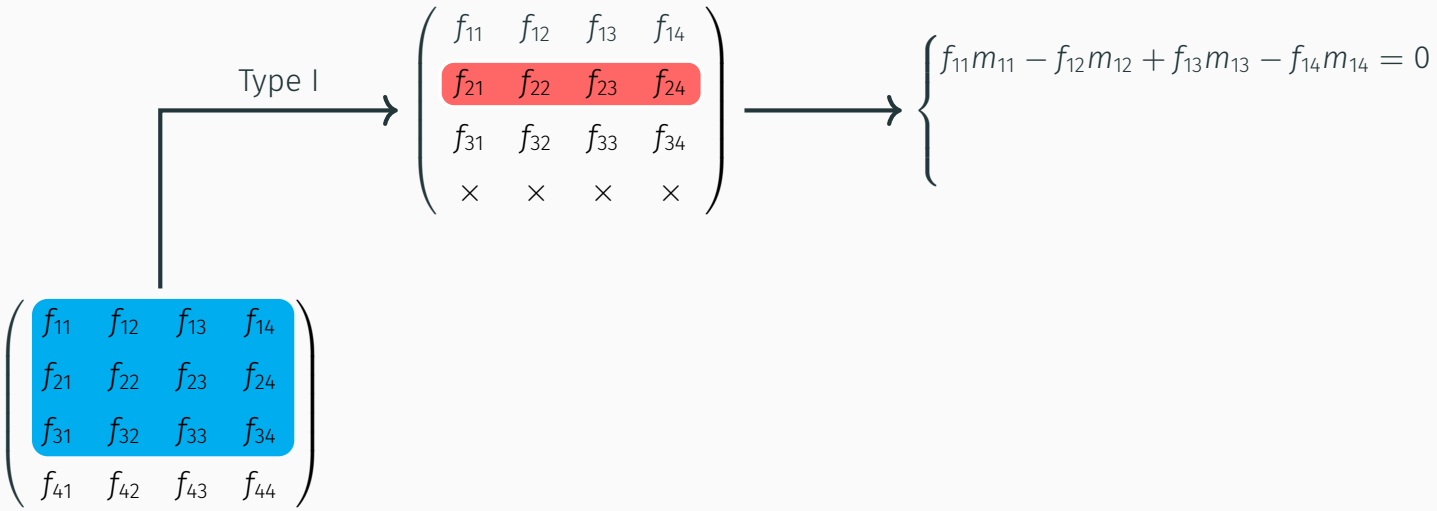

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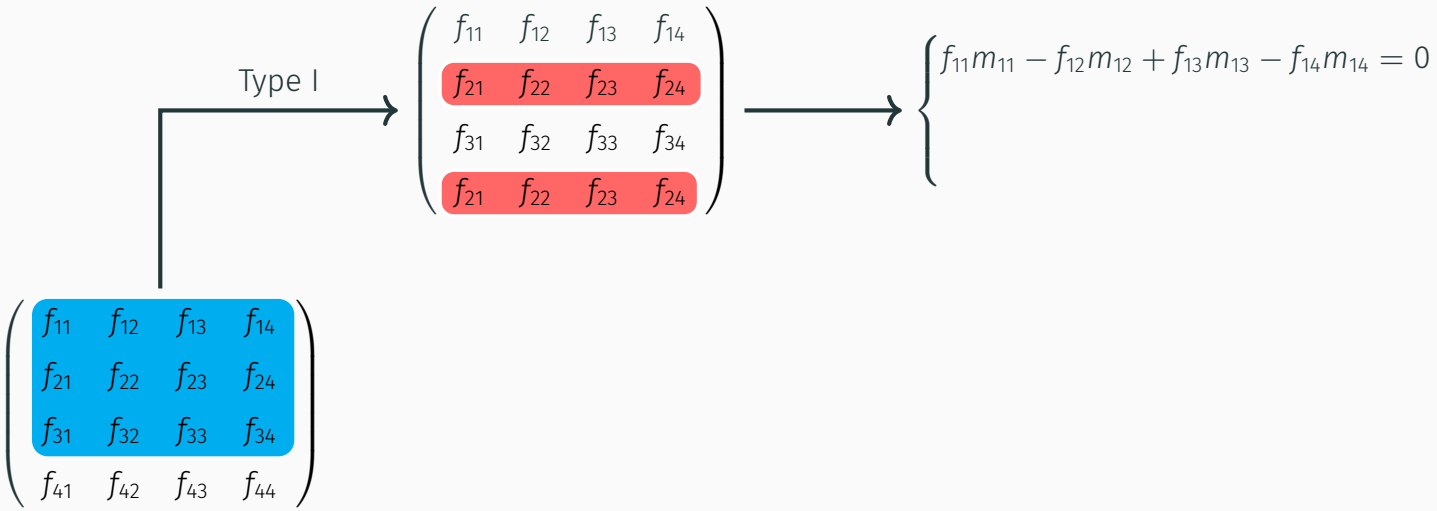

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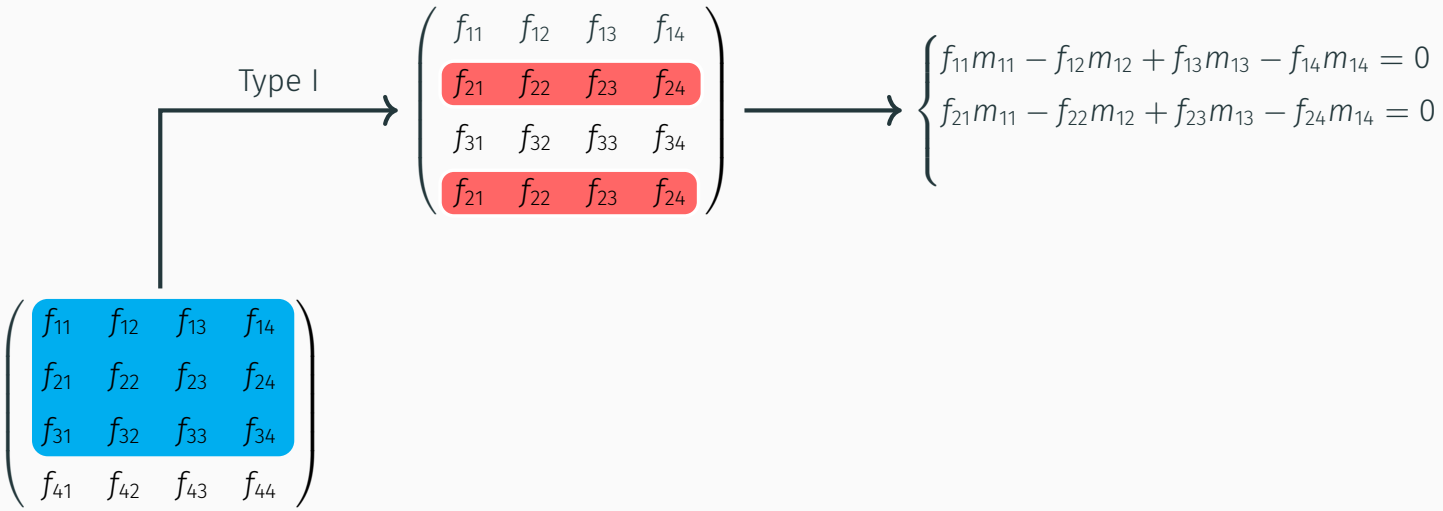

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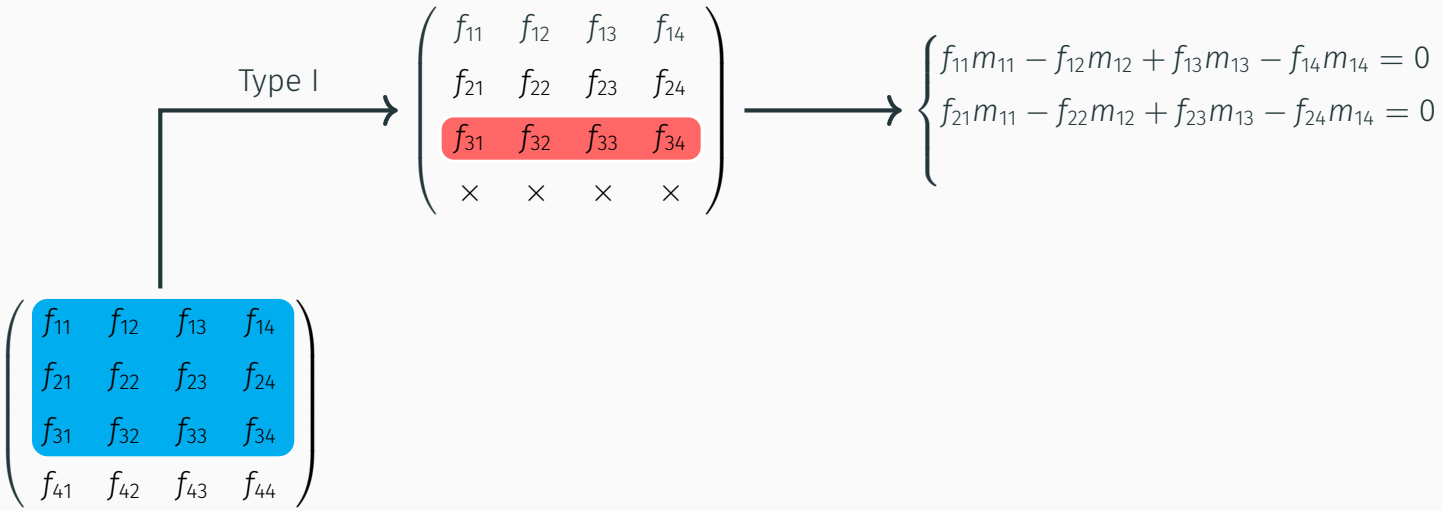

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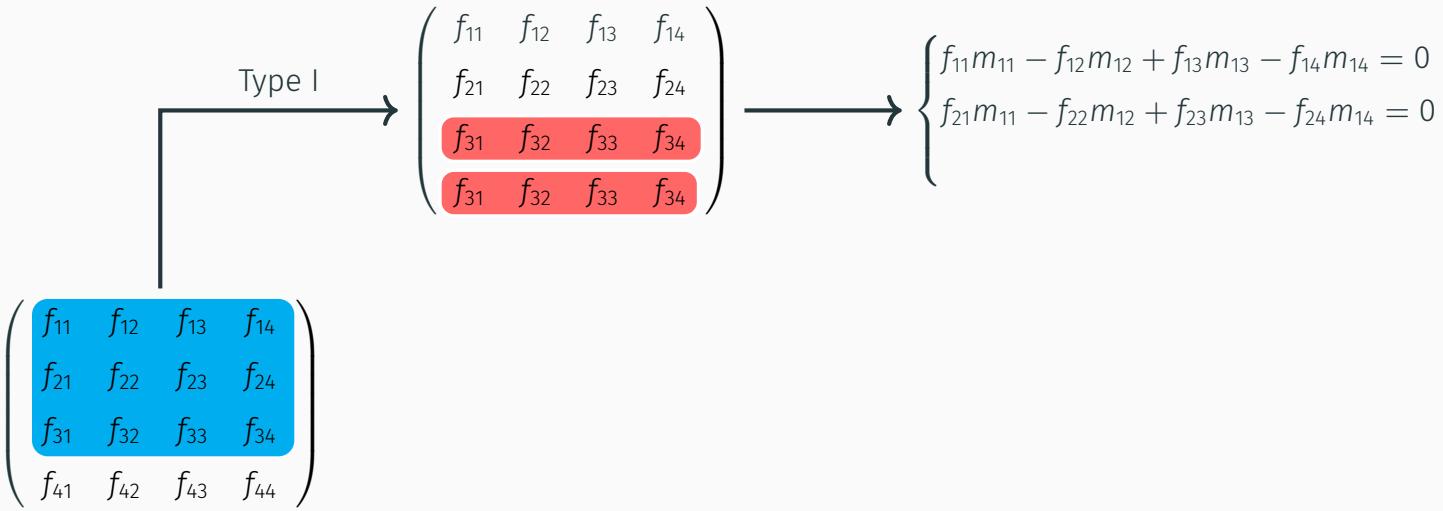

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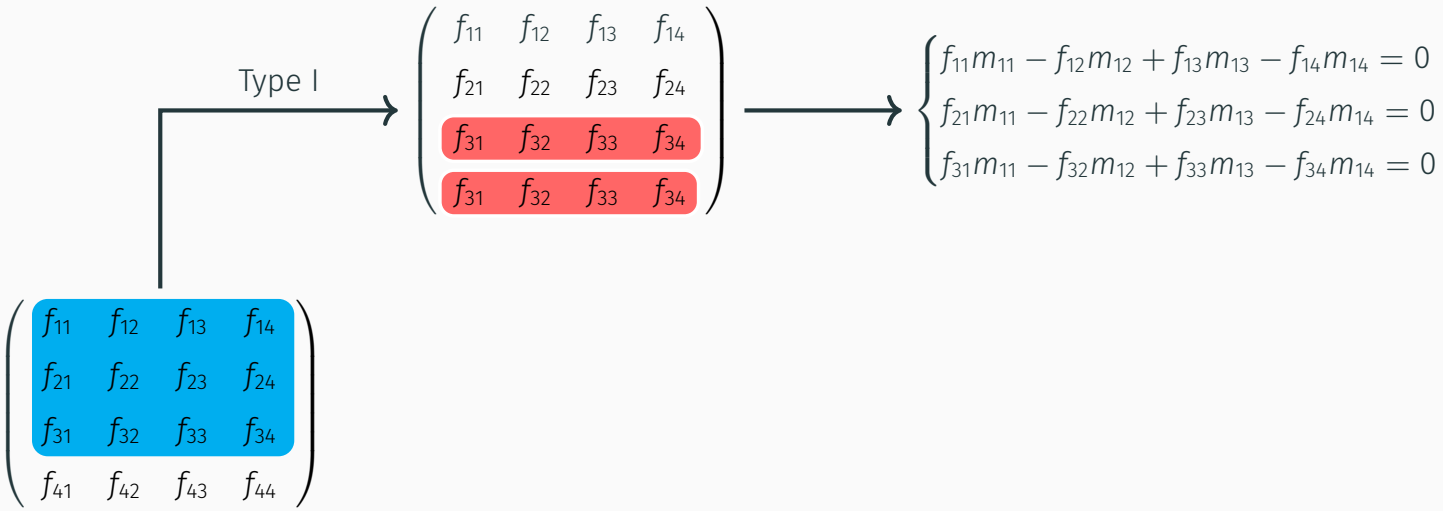

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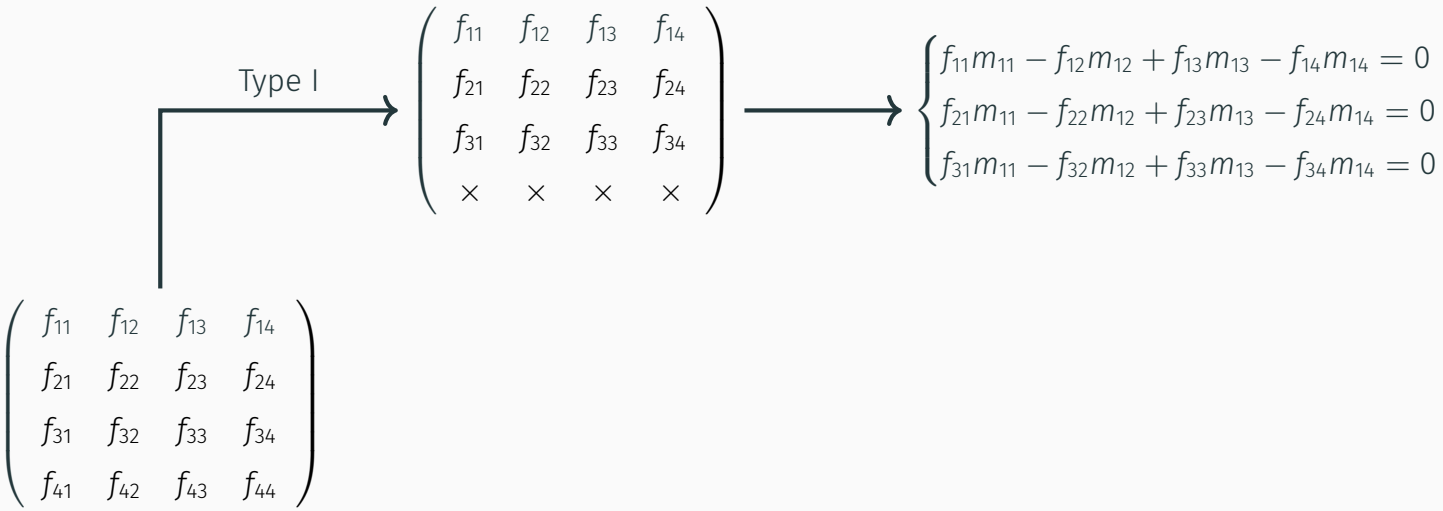

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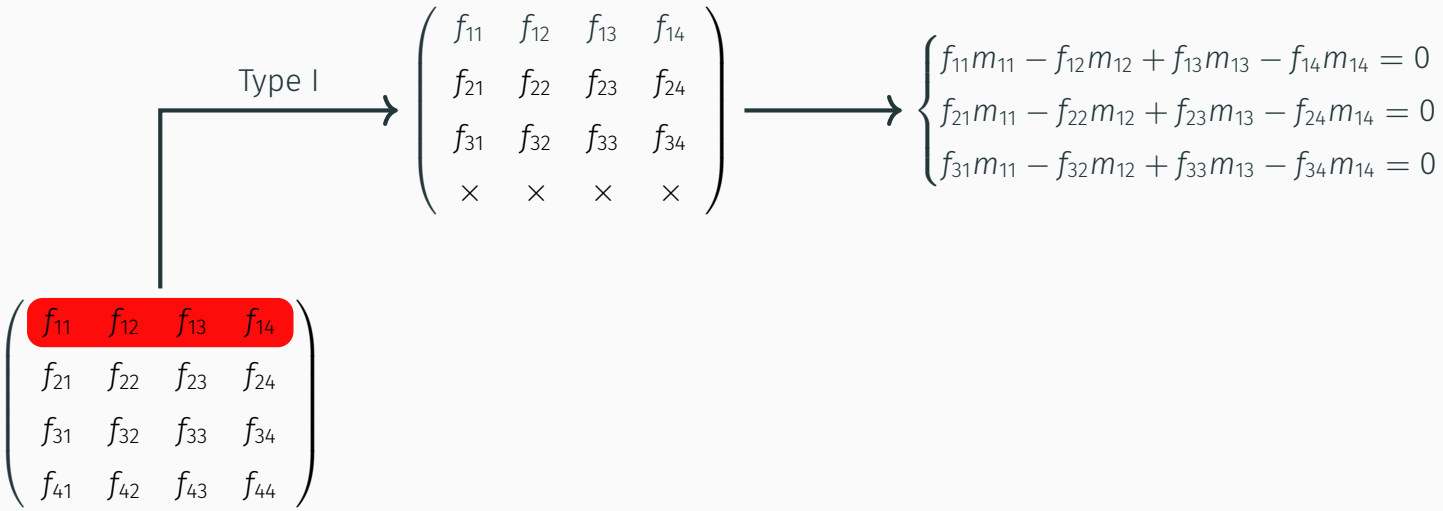

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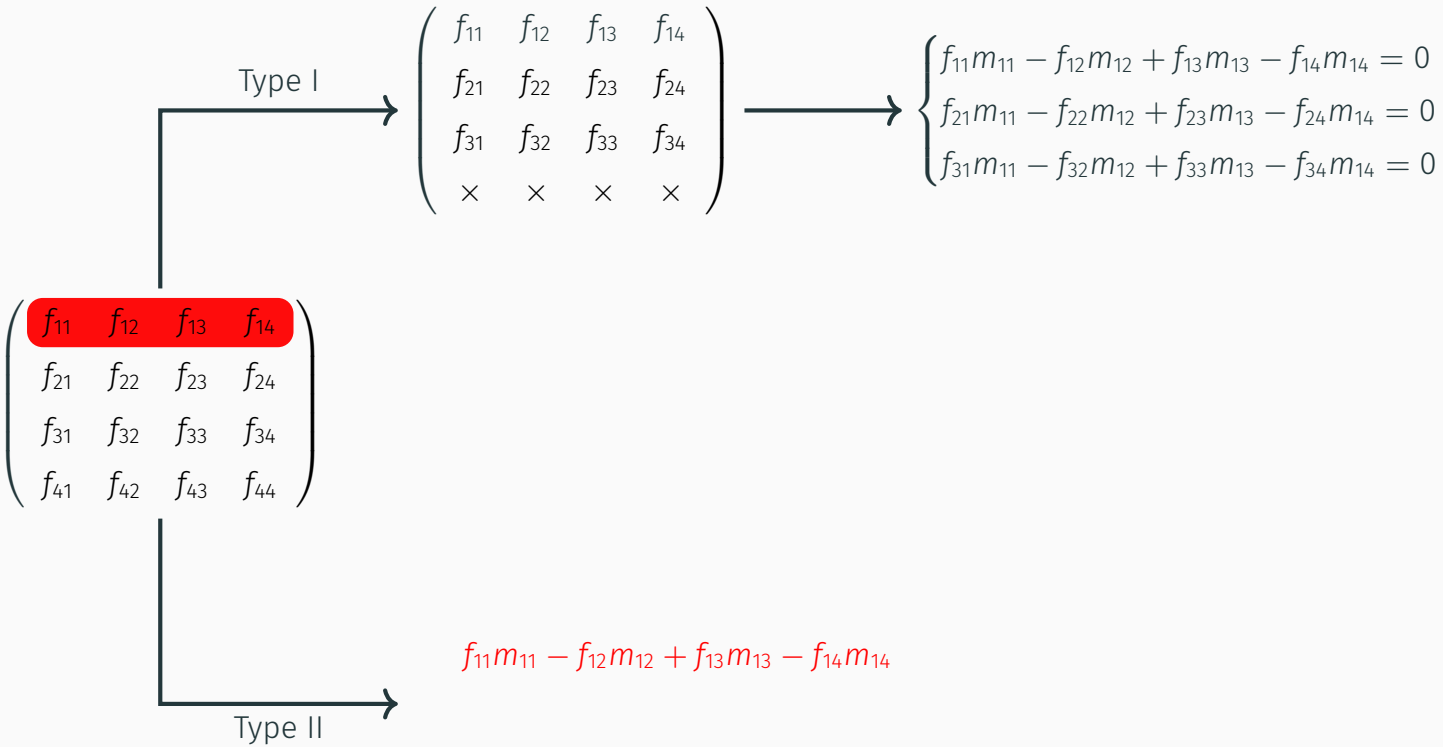

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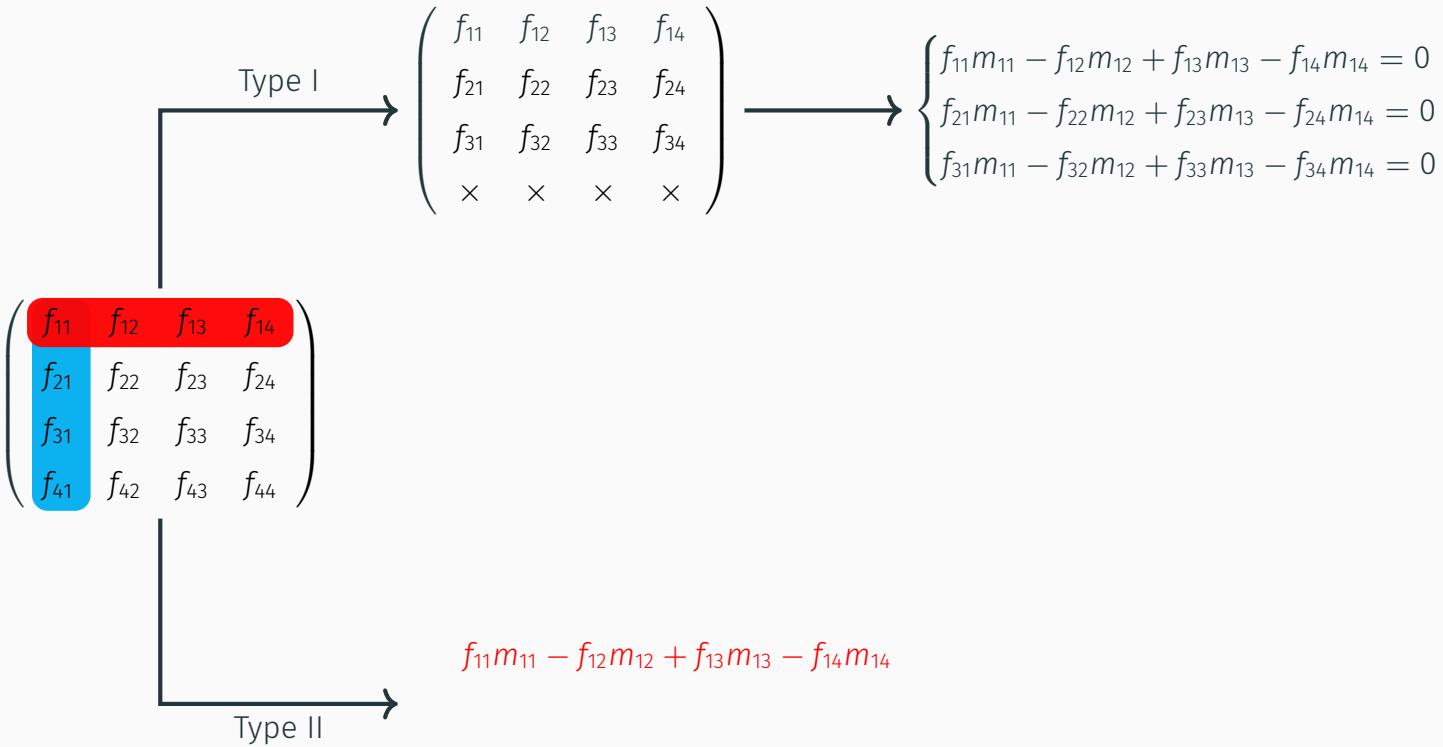

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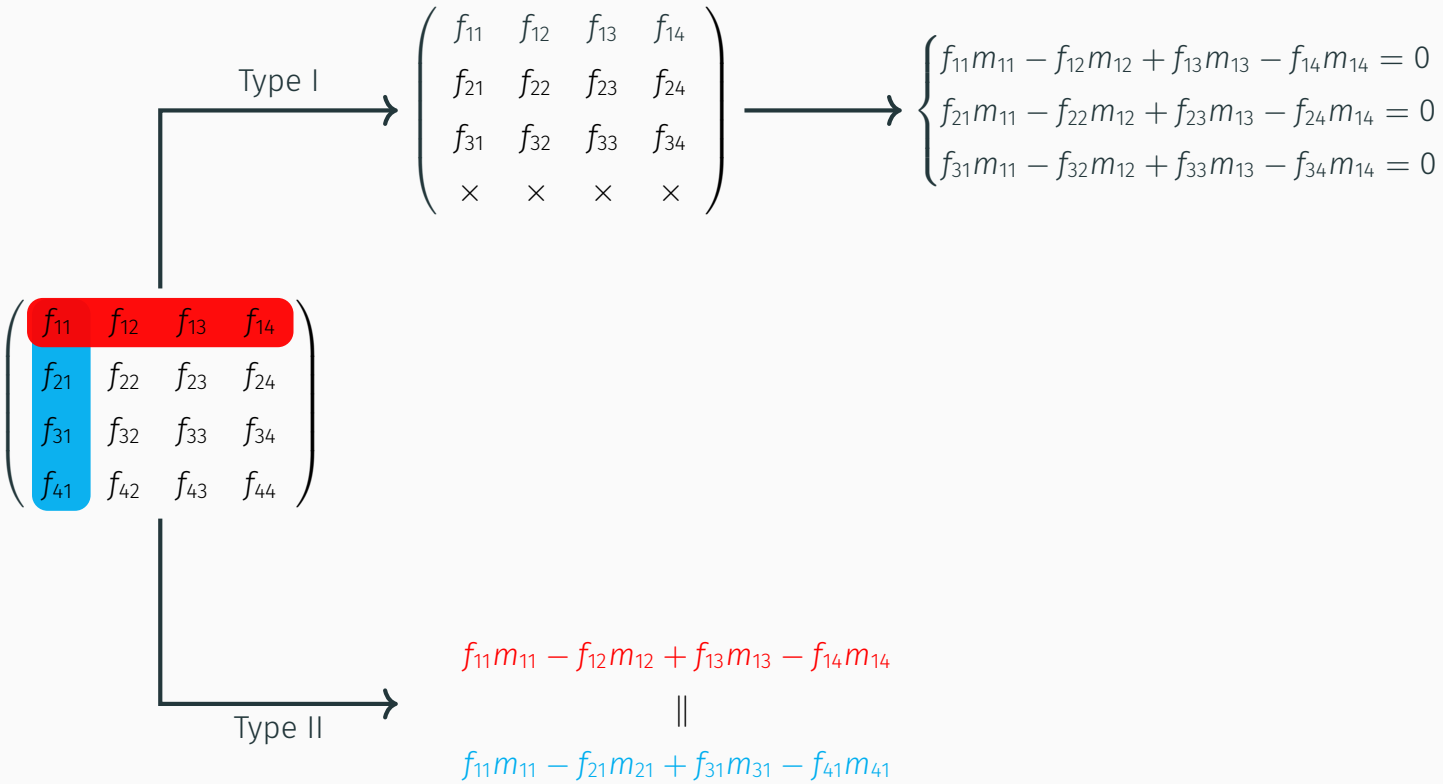

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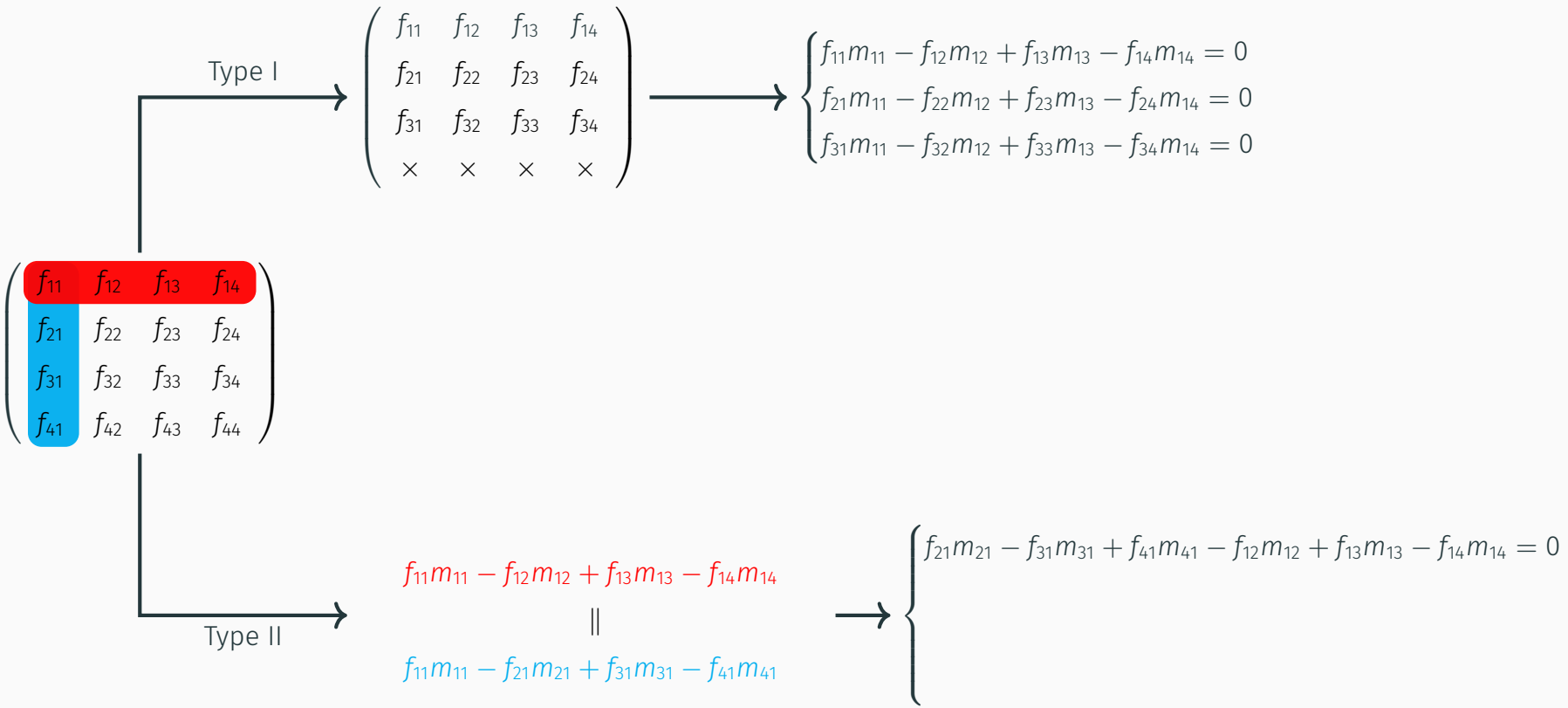

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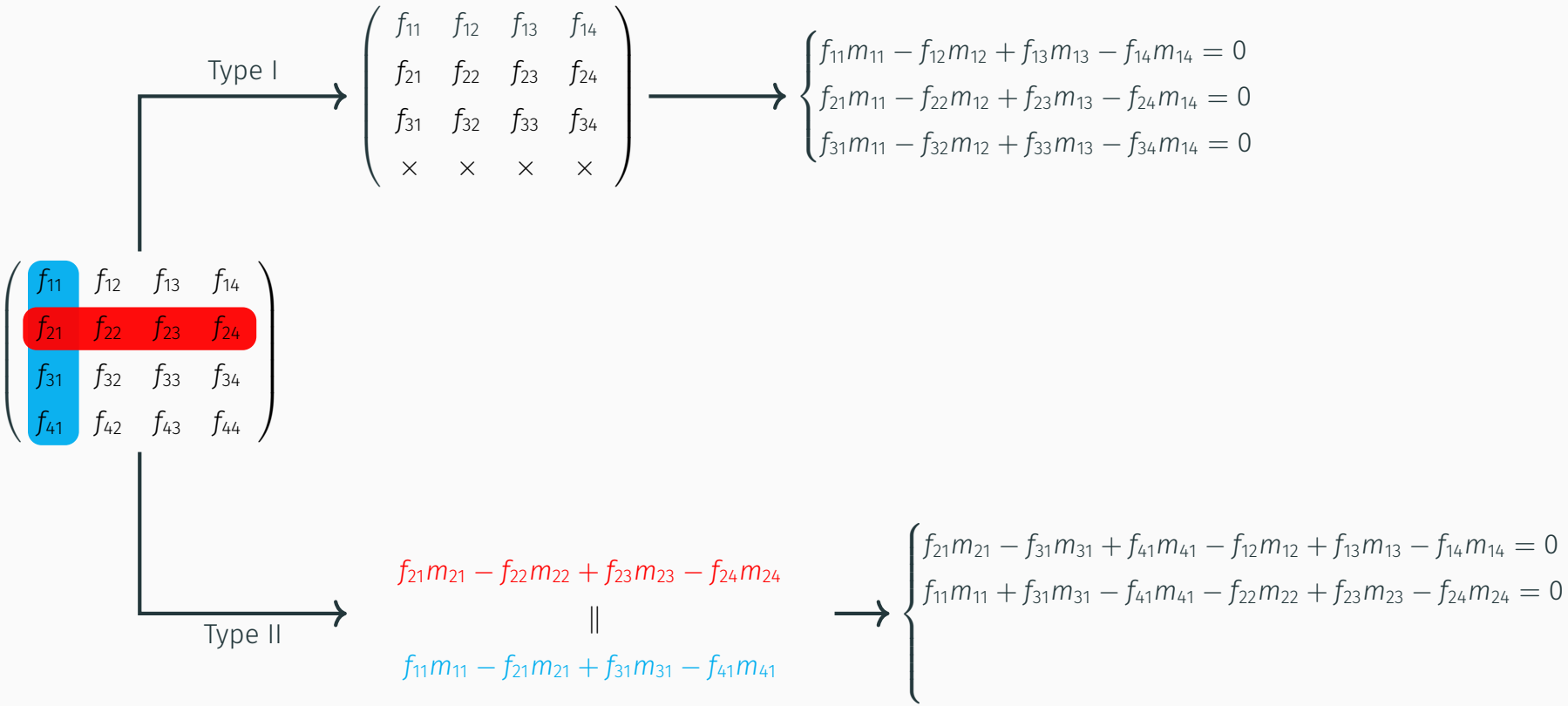

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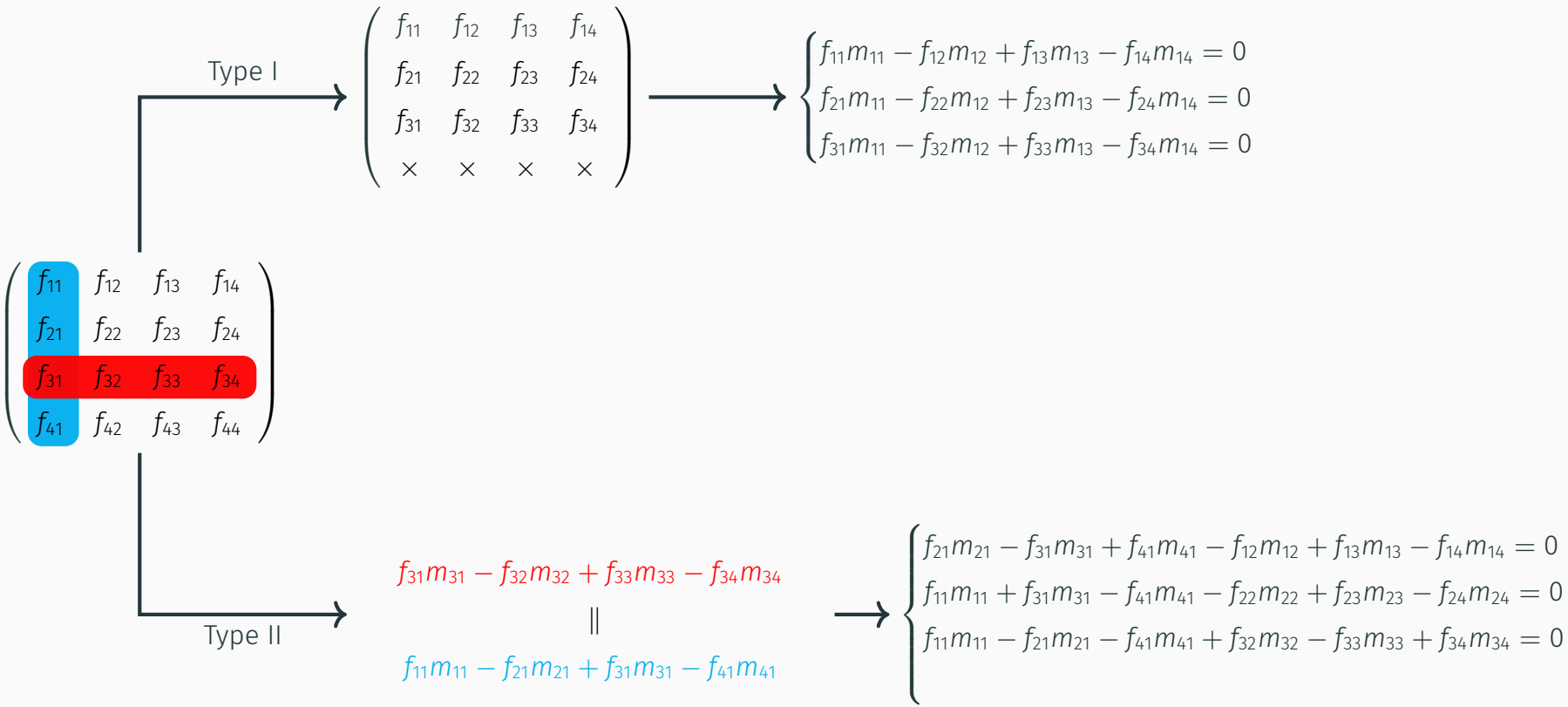

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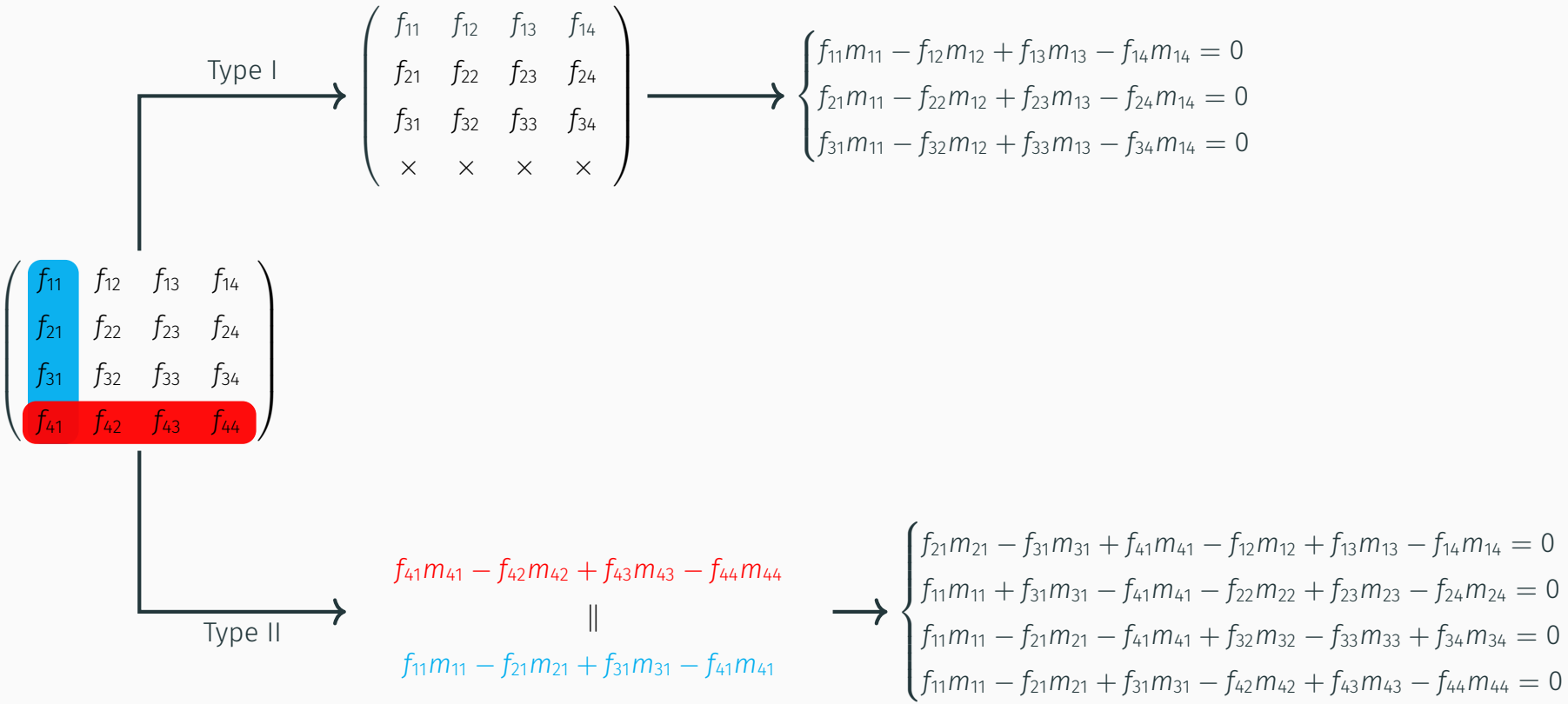

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$$\begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ f_{41} & f_{42} & f_{43} & f_{44} \end{pmatrix}$$

Type I

$$\begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ \times & \times & \times & \times \end{pmatrix} \longrightarrow \begin{cases} f_{11}m_{11} - f_{12}m_{12} + f_{13}m_{13} - f_{14}m_{14} = 0 \\ f_{21}m_{11} - f_{22}m_{12} + f_{23}m_{13} - f_{24}m_{14} = 0 \\ f_{31}m_{11} - f_{32}m_{12} + f_{33}m_{13} - f_{34}m_{14} = 0 \end{cases}$$

Type II

$$f_{41}m_{41} - f_{42}m_{42} + f_{43}m_{43} - f_{44}m_{44}$$
$$\| $$
$$f_{11}m_{11} - f_{21}m_{21} + f_{31}m_{31} - f_{41}m_{41}$$

$$\longrightarrow \begin{cases} f_{21}m_{21} - f_{31}m_{31} + f_{41}m_{41} - f_{12}m_{12} + f_{13}m_{13} - f_{14}m_{14} = 0 \\ f_{11}m_{11} + f_{31}m_{31} - f_{41}m_{41} - f_{22}m_{22} + f_{23}m_{23} - f_{24}m_{24} = 0 \\ f_{11}m_{11} - f_{21}m_{21} - f_{41}m_{41} + f_{32}m_{32} - f_{33}m_{33} + f_{34}m_{34} = 0 \\ f_{11}m_{11} - f_{21}m_{21} + f_{31}m_{31} - f_{42}m_{42} + f_{43}m_{43} - f_{44}m_{44} = 0 \end{cases}$$

Theorem ([Kurano, 1989])

The syzygies between the $(r+1)$-minors of M are generated by the syzygies between the $(r+1)$ minors of the $(r+2) \times (r+2)$ submatrices of M.

New F_5 algorithms - the general case

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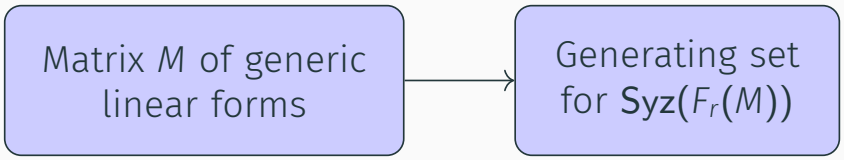
Matrix M of generic linear forms

New F_5 algorithms - the general case



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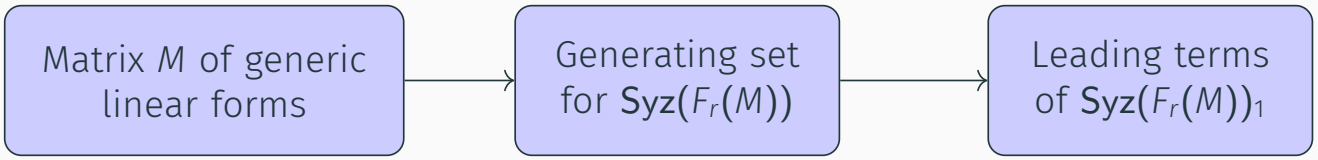

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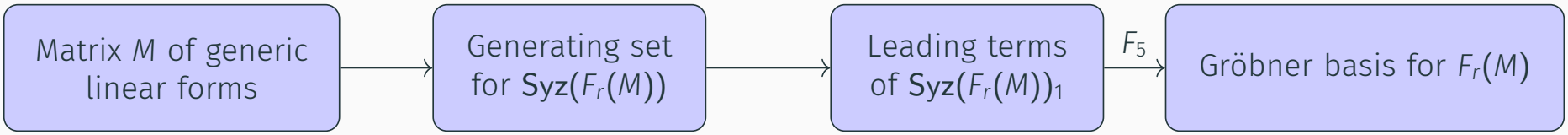

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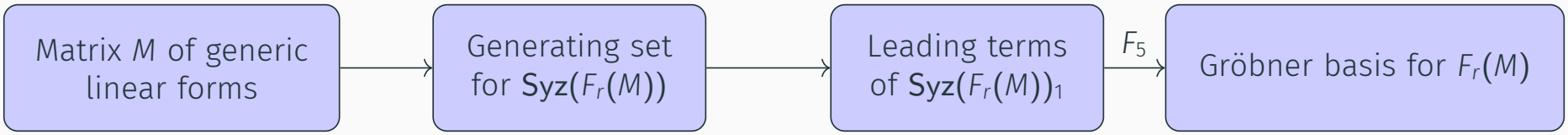

$$\#\,\mathsf{Syz}(F_r(M)) = \binom{n}{r+2}^2 \left(\frac{2(r+2)(r+1)}{n-r-1} + 2r + 2\right).$$

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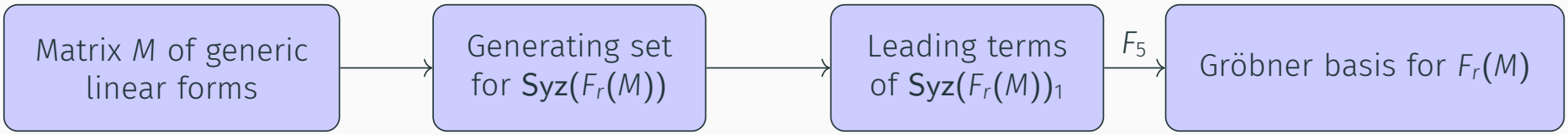

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Theorem ([Eagon, Hochster, 1971])

$F_r(M)$ has a free resolution of length $(n-r)^2$.

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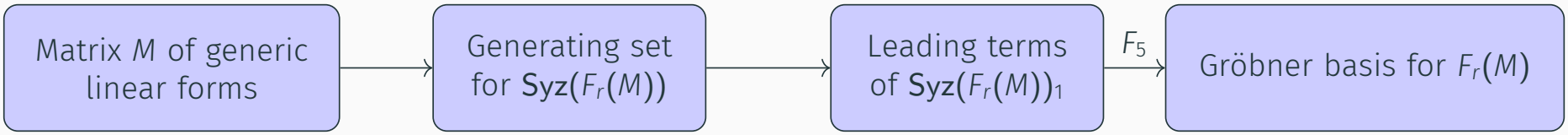

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$$\mathsf{Syz}_k(F_r(M)) \neq 0$$

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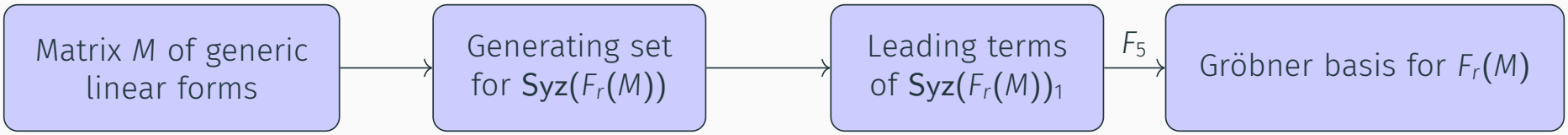

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$F_r(M)$ has a free resolution of length $(n-r)^2$.

$\mathsf{Syz}_k(F_r(M)) \neq 0$ for $1 < k < (n-r)^2$. $\Longrightarrow$ Cannot efficiently compute a Gröbner basis for $\mathsf{Syz}(F_r(M))$

New F_5 algorithms - the case $r = n - 2$

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Matrix M of generic linear forms

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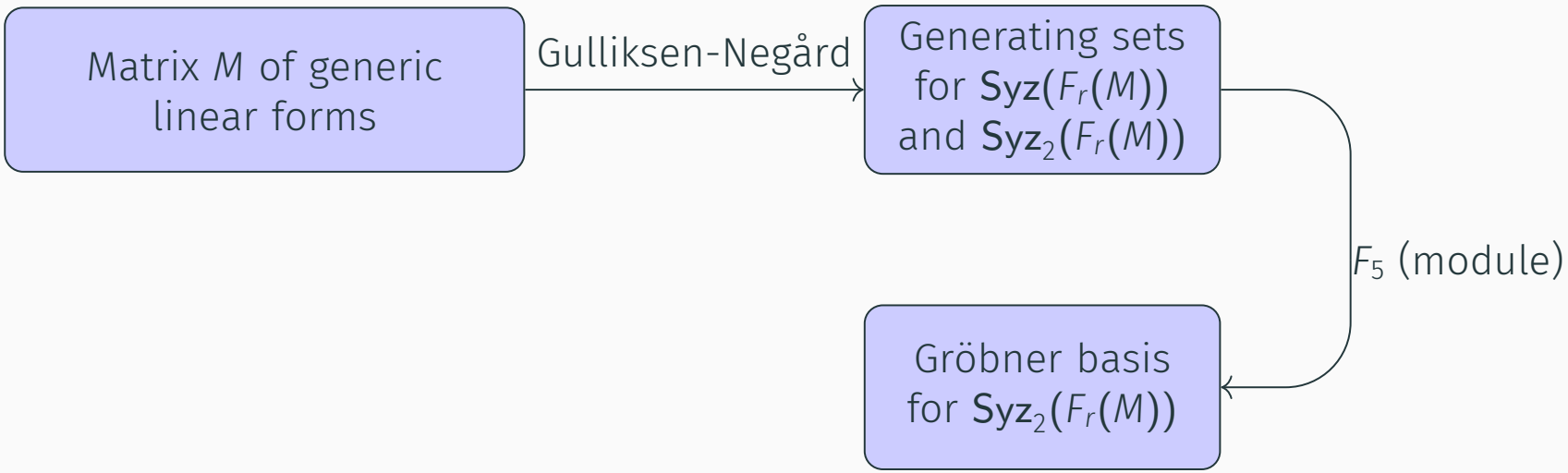

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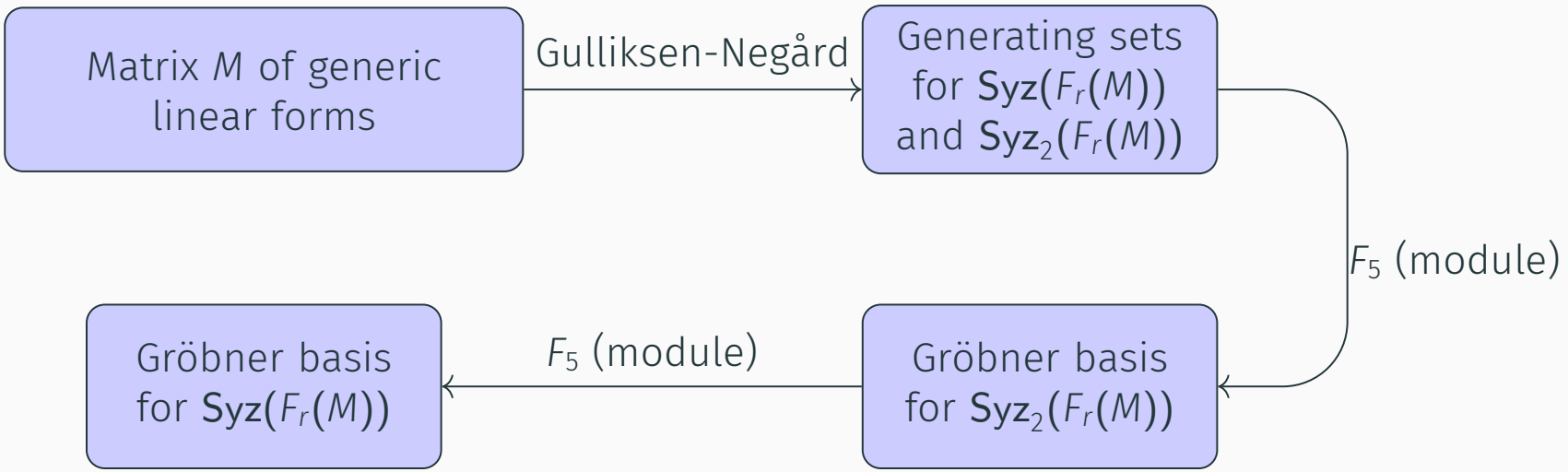

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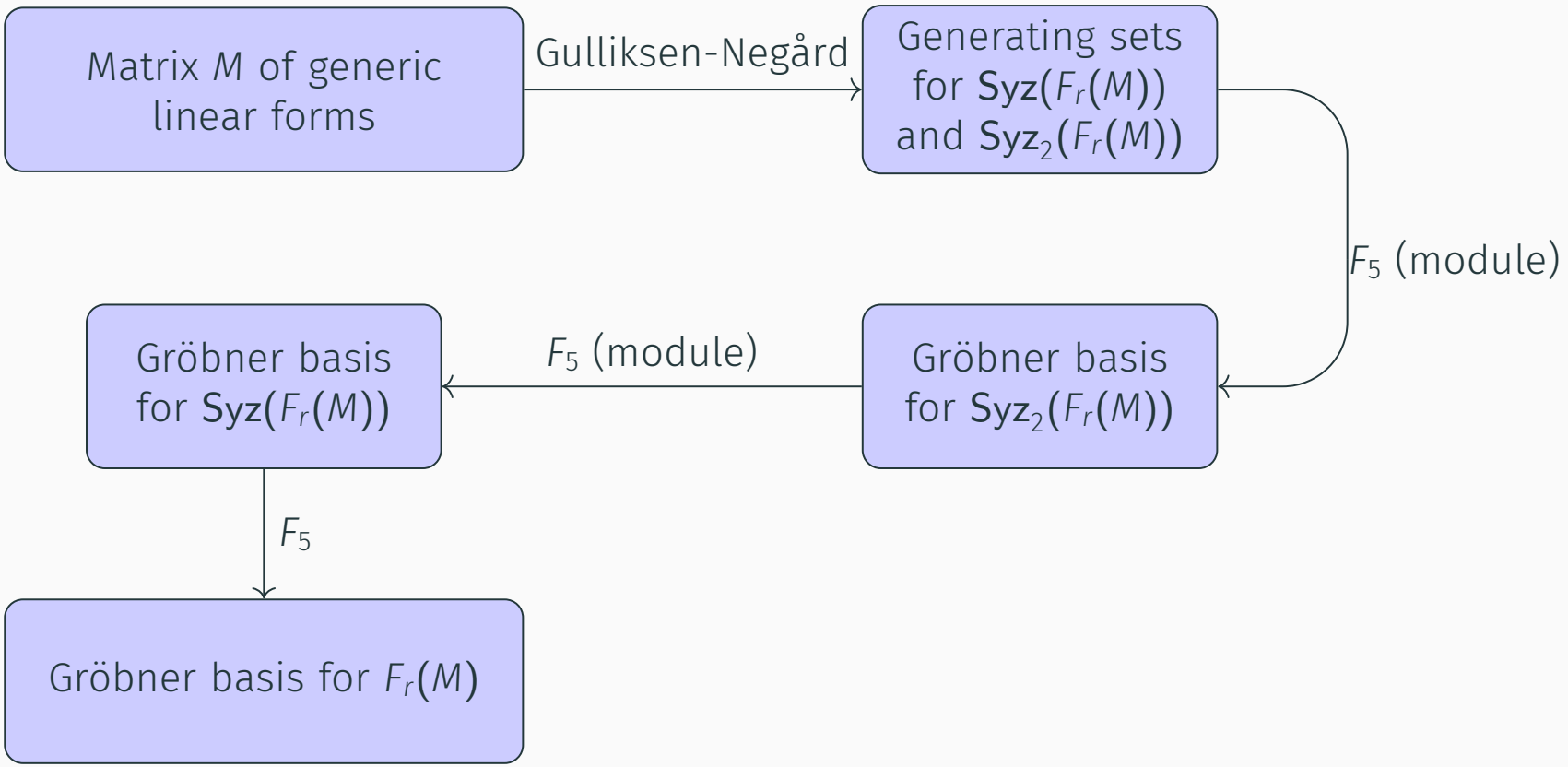

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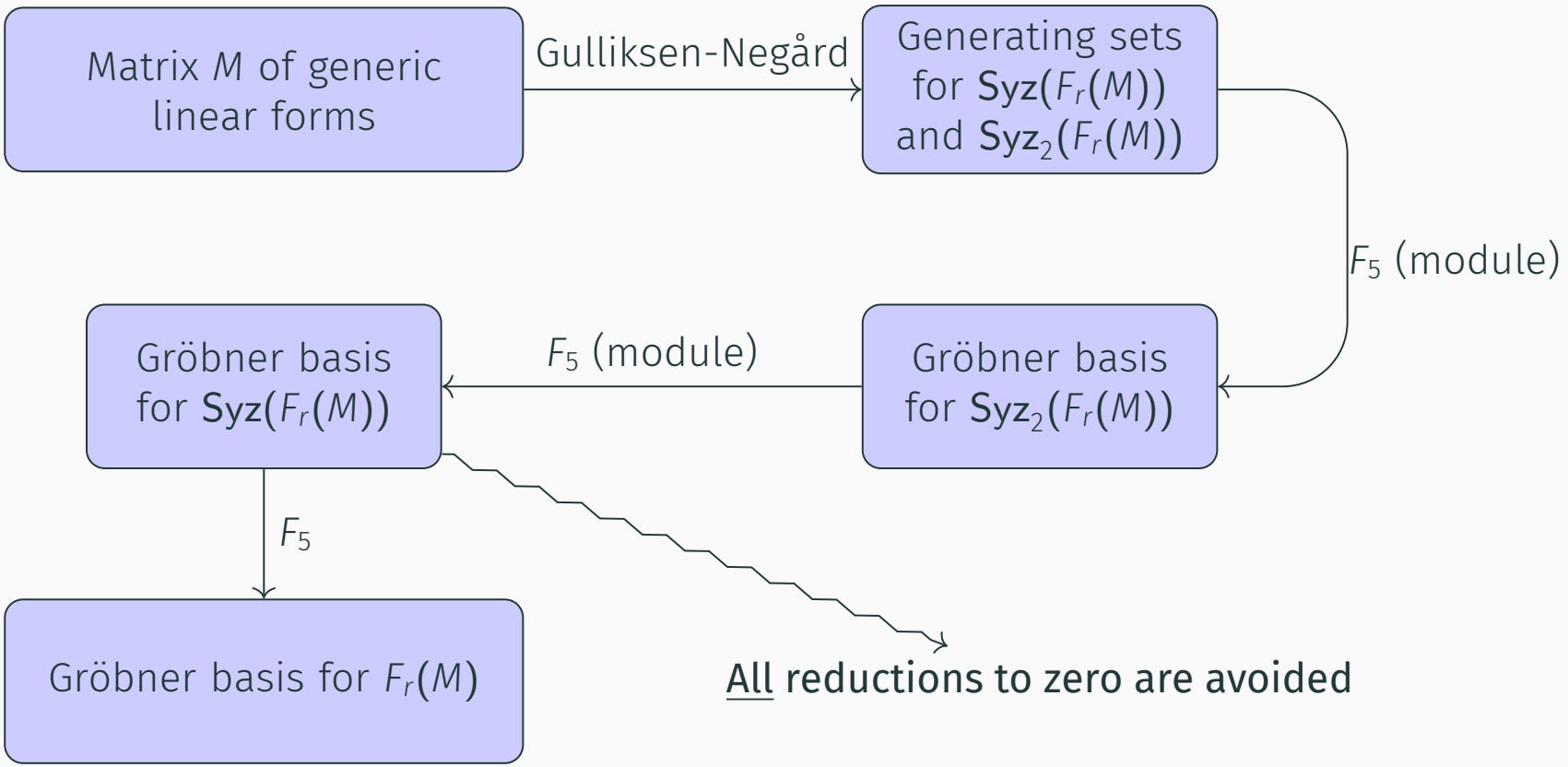

A complexity analysis in the case $r = n - 2$

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Gulliksen-
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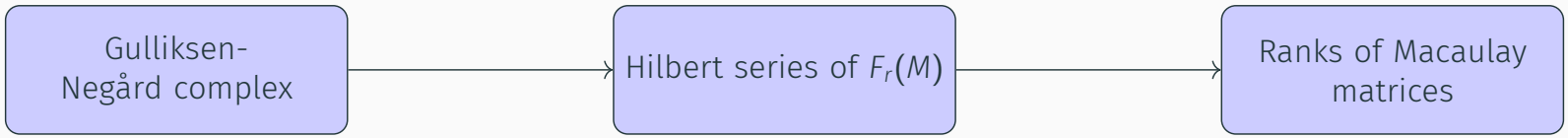

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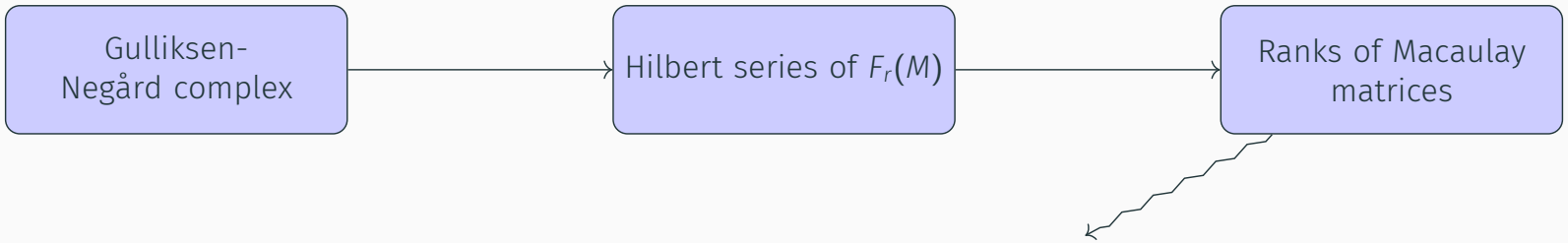

Theorem ([G., Neiger, Safey, 2023])

Let M be a matrix of generic linear forms in four variables. The complexity of computing a grevlex-Gröbner basis for $F_r(M)$ is in

$$O\left(\left(\sum_{d=n-1}^{2n-3} \frac{(d^2+(-2n+4)d+4n^2-4n+3)(2+d-n)}{3}\right)^{\omega-1}\binom{2n+1}{5}\right).$$

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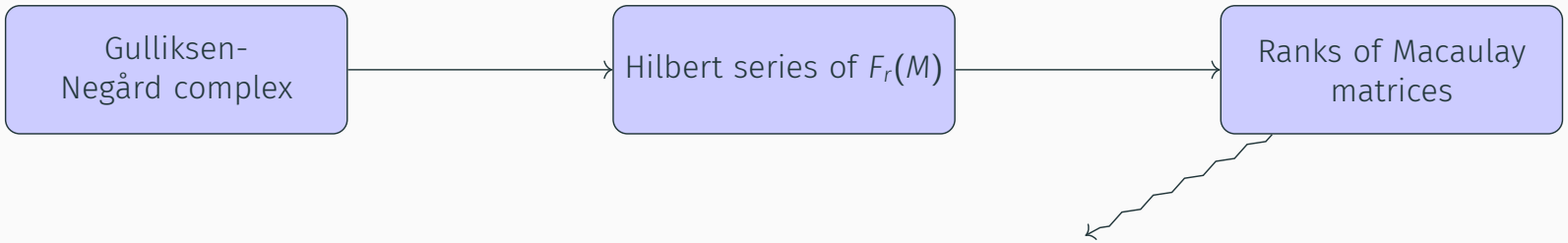

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Asymptotically:

[Faugère, Safey, Spaenlehauer, 2013]

$$O\left(n^{5\omega+2}\right)$$

[G., Neiger, Safey El Din, 2023]

$$O\left(n^{4\omega-1}\right)$$

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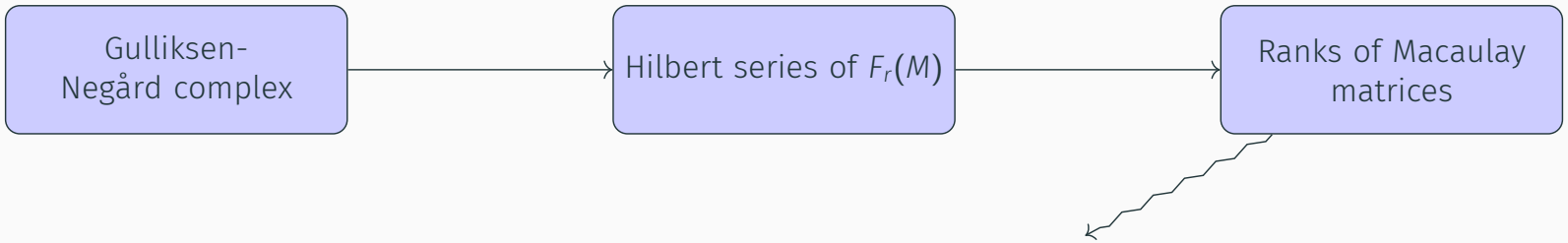

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$$O\left(n^{4\omega-1}\right)$$

Refined further to $O(n^{2\omega+3})$ and established lower bound $\Omega(n^6)$.

Experimental results

n	r	k	D	d	rank	Std. F_5	Det. F_5
8	6	4	13	7	64	64	64
				8	130	256	130
				9	200	322	200
				10	276	385	276
				11	360	471	360
				12	454	559	454
				13	560	650	560
9	7	4	15	8	81	81	81
				9	164	324	164
				10	251	401	251
				11	344	486	344
				12	445	584	445
				13	556	675	556
				14	679	813	679
				15	816	931	816

n	r	k	D	d	rank	Std. F_5	Det. F_5
4	1	9	4	2	36	36	36
				3	164	324	164
				4	495	582	582
5	2	9	7	3	100	100	100
				4	450	900	450
				5	1278	1956	1956
				6	3002	3546	3546
				7	6435	6685	6685
6	3	9	6	4	225	225	225
				5	1017	2025	1017
				6	2838	4715	4715
7	4	9	6	5	441	441	441
				6	2009	3969	2009

n	r	k	D	d	rank	Std. F_5	Det. F_5
5	1	16	4	2	100	100	100
				3	800	1600	800
				4	3875	4662	4662
6	2	16	4	3	400	400	400
				4	3250	6400	3250

$k =$ number of variables.
$D =$ highest degree appearing in the (reduced) grevlex Gröbner basis for $F_r(M)$.

- When $r = n - 2$, all Macaulay matrices are full rank.
- When $r < n - 2$, the Macaulay matrix in degree $r + 2$ is full rank.
- Many reductions to zero remain in higher degrees.

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5	2	9	7	3	100	100	100
				4	450	900	450
				5	1278	1956	1956
				6	3002	3546	3546
				7	6435	6685	6685
6	3	9	6	4	225	225	225
				5	1017	2025	1017
				6	2838	4715	4715

n	r	k	D	d	rank	Std. F_5	Det. F_5
5	1	16	4	2	100	100	100
				3	800	1600	800
				4	3875	4662	4662
6	2	16	4	3	400	400	400
				4	3250	6400	3250

$\sim 30\%$ of reductions to zero removed in general case

$k =$ number of variables.

$D =$ highest degree appearing in the (reduced) grevlex Gröbner basis for $F_r(M)$.

- When $r = n - 2$, all Macaulay matrices are full rank.
- When $r < n - 2$, the Macaulay matrix in degree $r + 2$ is full rank.
- Many reductions to zero remain in higher degrees.

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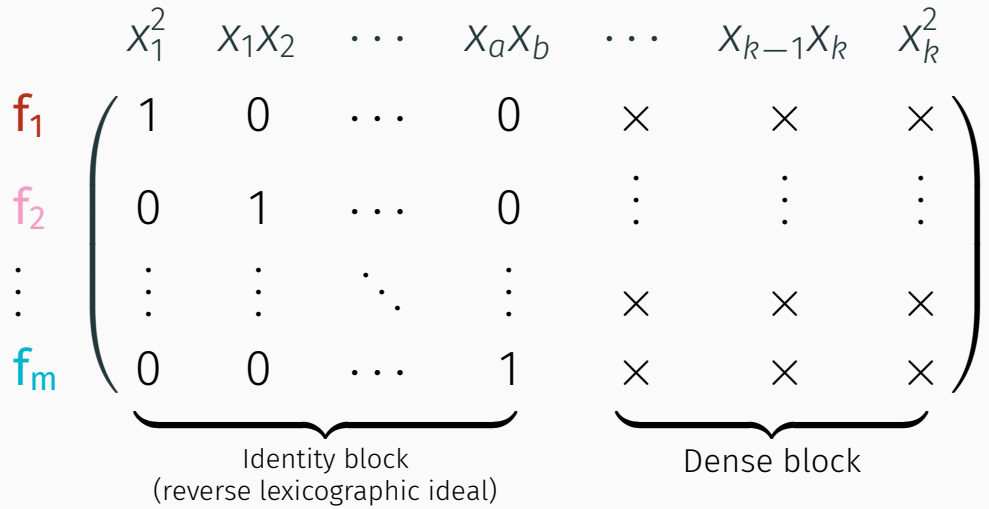
- Efficient implementations of new algorithms.

Thanks. Questions?

Sharper complexity bounds



Sharper complexity bounds

$$
\begin{array}{c}
 \\
\mathsf{f}_1 \\
\mathsf{f}_2 \\
\vdots \\
\mathsf{f}_m
\end{array}
\begin{array}{c}
\begin{array}{cccccccc}
x_1^2 & x_1x_2 & \cdots & x_ax_b & \cdots & x_{k-1}x_k & x_k^2
\end{array} \\
\begin{pmatrix}
1 & 0 & \cdots & 0 & \times & \times & \times \\
0 & 1 & \cdots & 0 & \vdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \times & \times & \times \\
0 & 0 & \cdots & 1 & \times & \times & \times
\end{pmatrix} \\
\underbrace{\qquad\qquad\qquad\qquad}_{\text{Identity block}} \quad \underbrace{\qquad\qquad\qquad}_{\text{Dense block}} \\
\text{(reverse lexicographic ideal)}
\end{array}
$$

$$
\begin{array}{c}
 \\
x_1\mathsf{f}_1 \\
x_2\mathsf{f}_1 \\
\vdots \\
x_1\mathsf{f}_2 \\
\vdots \\
x_k\mathsf{f}_m
\end{array}
\begin{array}{c}
\begin{array}{cccccccc}
x_1^3 & x_1^2x_2 & \cdots & x_kx_ax_b & \cdots & x_{k-1}x_k^2 & x_k^3
\end{array} \\
\begin{pmatrix}
1 & 0 & \cdots & 0 & \times & \times & \times \\
0 & 1 & \cdots & 0 & \vdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 1 & \cdots & 0 & \vdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \times & \times & \times \\
0 & 0 & \cdots & 1 & \times & \times & \times
\end{pmatrix}
\end{array}
$$

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$$
\begin{array}{c}
 & \begin{array}{ccccccc} x_1^2 & x_1x_2 & \cdots & x_ax_b & \cdots & x_{k-1}x_k & x_k^2 \end{array} \\
\begin{array}{c} f_1 \\ f_2 \\ \vdots \\ f_m \end{array} & \left(\begin{array}{ccccccc} 1 & 0 & \cdots & 0 & \times & \times & \times \\ 0 & 1 & \cdots & 0 & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \times & \times & \times \\ 0 & 0 & \cdots & 1 & \times & \times & \times \end{array}\right) \\
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 & \begin{array}{ccccccc} x_1^3 & x_1^2x_2 & \cdots & x_kx_ax_b & \cdots & x_{k-1}x_k^2 & x_k^3 \end{array} \\
\begin{array}{c} x_1f_1 \\ x_2f_1 \\ \vdots \\ x_1f_2 \\ \vdots \\ x_kf_m \end{array} & \left(\begin{array}{ccccccc} 1 & 0 & \cdots & 0 & \times & \times & \times \\ 0 & \boxed{1} & \cdots & 0 & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \boxed{1} & \cdots & 0 & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \times & \times & \times \\ 0 & 0 & \cdots & 1 & \times & \times & \times \end{array}\right)
\end{array}
$$

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\begin{array}{c}
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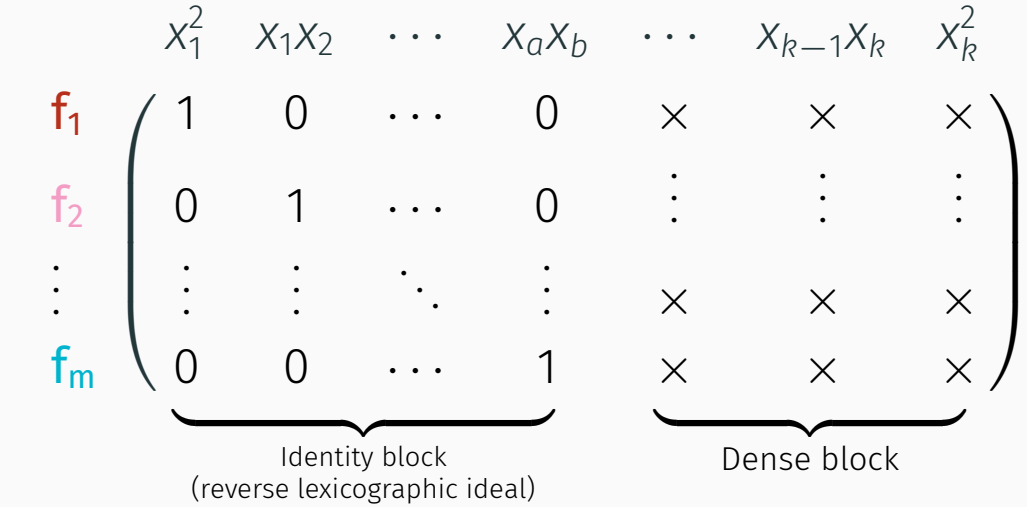
\vdots & \vdots & \ddots & \vdots & \times & \times & \times \\
0 & 0 & \cdots & 1 & \times & \times & \times
\end{pmatrix}
\end{array}
$$

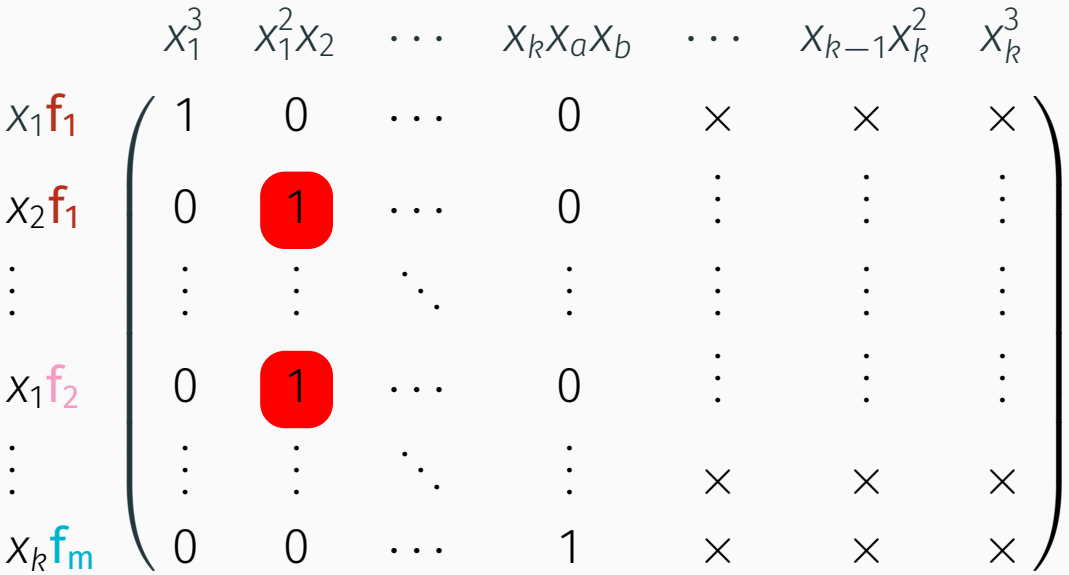
"Collisions" in Macaulay matrices

↕

New GB elements **or** reductions to zero

Sharper complexity bounds





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New GB elements **or** reductions to zero

Revlex+Hilbert series

⇓

Exact size of GB