Optimized Gröbner basis algorithms for maximal determinantal ideals and critical point computations

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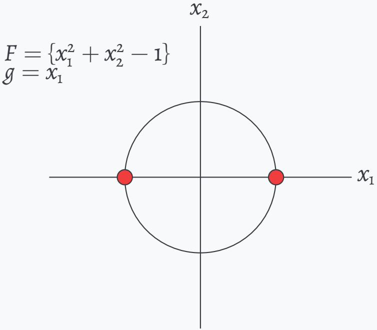

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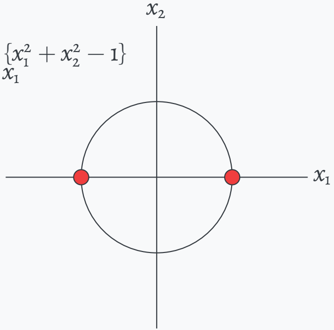

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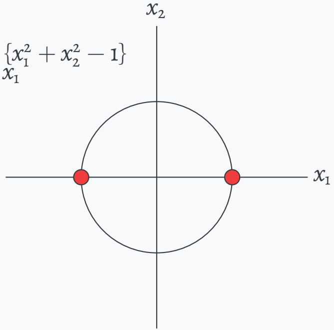

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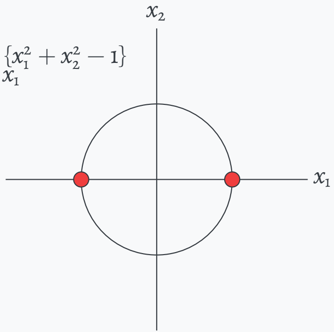

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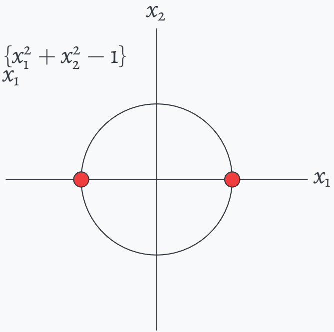

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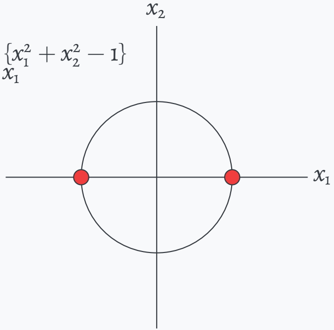

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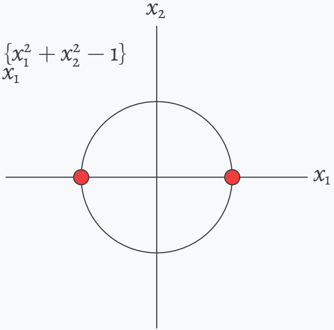

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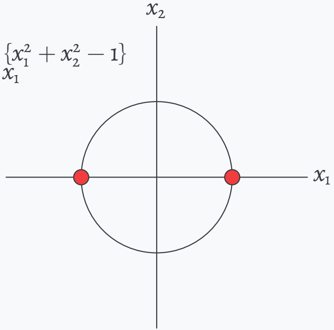

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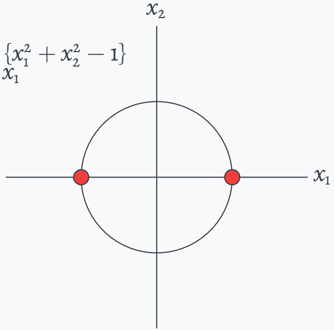

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Notation

$F_{p+1}(M) =$ system of $(p+1)$-minors of M

$$\mathcal{J}_{p+1}(M) = \langle F_{p+1}(M) \rangle$$

Gröbner bases

$F \subseteq \mathbb{k}[x_1, \ldots, x_n]$ a polynomial system.

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Definition (Gröbner bases)

A $\succ$-*Gröbner basis* is a finite generating set G for $\langle F \rangle$ such that

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For zero-dimensional systems:

$$\max_{f \in F}\{\deg f\}^{O(\#\text{ of variables})}$$ **[Lazard, 1983]**

Macaulay matrices - linearization

Assume F homogeneous.

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$$\begin{cases} f_1 = 2x^2 + 11xy - y^2 \\ f_2 = 4x^2 + xy - 2y^2 \\ f_3 = -6x^2 - xy + y^2 \end{cases}$$

$$\begin{array}{cccc} x^3 & x^2y & xy^2 & y^3 \end{array}$$
$$\begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & \end{pmatrix}$$

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$\mathbf{f}_1 \xrightarrow{\cdot x} x\mathbf{f}_1$, $\mathbf{f}_1 \xrightarrow{\cdot y} y\mathbf{f}_1$, $\mathbf{f}_2 \xrightarrow{\cdot x} x\mathbf{f}_2$

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The rows of the echelonization of the Macaulay matrix of F in degree d form the elements of degree d of a $\succ$-Gröbner basis for F.

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Theorem (Macaulay bound) **[Lazard, 1983]**

The maximum degree of a polynomial in the **grevlex** Gröbner basis of a generic polynomial system $f_1, \ldots, f_m$ is

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Theorem **[Spaenlehauer, 2013]**

$F = (f_1, \dots, f_p) \subseteq \mathbb{k}[x_1, \dots, x_n]$, $g \in \mathbb{k}[x_1, \dots, x_n]$ all homogeneous of degree δ. Generically, the number of arithmetic operations in $\mathbb{k}$ required to compute a grevlex GB of $\mathcal{I}(g, F) = \langle F \rangle + \mathcal{I}_{p+1}(\mathrm{jac}(g, F))$ is in

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This upper bound is not very sharp since the Macaulay matrices are not full rank!

The F_5 algorithm

[Faugère, 2002]

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$$
\begin{aligned}
f_1 &= 5x^2 + 5xy + 3y^2 + 5xz + 5yz + 6z^2 \\
f_2 &= 2x^2 + xy + 4y^2 + 2xz + 4z^2 \\
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\end{aligned}
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$$\widetilde{\mathcal{M}}_2$$

	x^2	xy	y^2	xz	yz	z^2
f_1	1	1	2	1	1	4
f_2	0	1	0	0	2	4
f_3	0	0	1	2	0	4

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$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
x^2f_1	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
xyf_1	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
y^2f_1	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
xzf_1	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
yzf_1	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
z^2f_1	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
x^2f_2	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
xyf_2	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
y^2f_2	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
xzf_2	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
yzf_2	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
z^2f_2	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
x^2f_3	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
xyf_3	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
y^2f_3	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
xzf_3	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
yzf_3	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
z^2f_3	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

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$$f_2 = 2x^2 + xy + 4y^2 + 2xz + 4z^2$$
$$f_3 = 4x^2 + xy + 4y^2 + 3xz + 5yz + 2z^2$$

$\widetilde{\mathcal{M}}_2$

	x^2	xy	y^2	xz	yz	z^2
f_1	1	1	2	1	1	4
f_2	0	1	0	0	2	4
f_3	0	0	1	2	0	4

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
x^2f_1	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
xyf_1	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
y^2f_1	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
xzf_1	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
yzf_1	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
z^2f_1	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
x^2f_2	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
xyf_2	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
y^2f_2	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
xzf_2	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
yzf_2	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
z^2f_2	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
x^2f_3	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
xyf_3	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
y^2f_3	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
xzf_3	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
yzf_3	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
z^2f_3	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

The F_5 algorithm

[Faugère, 2002]

Let $\mathbb{k} = \mathbb{F}_7$, $\succ =$ grevlex.

$$f_1 = 5x^2 + 5xy + 3y^2 + 5xz + 5yz + 6z^2$$
$$f_2 = 2x^2 + xy + 4y^2 + 2xz + 4z^2$$
$$f_3 = 4x^2 + xy + 4y^2 + 3xz + 5yz + 2z^2$$

$\widetilde{\mathcal{M}}_2$

	x^2	xy	y^2	xz	yz	z^2
f_1	1	1	2	1	1	4
f_2	0	1	0	0	2	4
f_3	0	0	1	2	0	4

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
x^2f_1	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
xyf_1	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
y^2f_1	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
xzf_1	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
yzf_1	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
z^2f_1	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
x^2f_2	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
xyf_2	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
y^2f_2	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
xzf_2	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
yzf_2	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
z^2f_2	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
x^2f_3	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
xyf_3	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
y^2f_3	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
xzf_3	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
yzf_3	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
z^2f_3	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

The F_5 algorithm [Faugère, 2002]

Let $\mathbb{k} = \mathbb{F}_7$, $\succ =$ grevlex.

$$f_1 = 5x^2 + 5xy + 3y^2 + 5xz + 5yz + 6z^2$$
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$$f_3 = 4x^2 + xy + 4y^2 + 3xz + 5yz + 2z^2$$

$\widetilde{\mathcal{M}}_2$

	x^2	xy	y^2	xz	yz	z^2
f_1	1	1	2	1	1	4
f_2	0	1	0	0	2	4
f_3	0	0	1	2	0	4

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
x^2f_1	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
xyf_1	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
y^2f_1	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
xzf_1	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
yzf_1	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
z^2f_1	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
x^2f_2	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
xyf_2	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
y^2f_2	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
xzf_2	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
yzf_2	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
z^2f_2	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
x^2f_3	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
xyf_3	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
y^2f_3	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
xzf_3	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
yzf_3	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
z^2f_3	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

The F_5 algorithm **[Faugère, 2002]**

Let $\mathbb{k} = \mathbb{F}_7$, $\succ =$ grevlex.

$$f_1 = 5x^2 + 5xy + 3y^2 + 5xz + 5yz + 6z^2$$
$$f_2 = 2x^2 + xy + 4y^2 + 2xz + 4z^2$$
$$f_3 = 4x^2 + xy + 4y^2 + 3xz + 5yz + 2z^2$$

$\widetilde{\mathcal{M}}_2$

	x^2	xy	y^2	xz	yz	z^2
f_1	1	1	2	1	1	4
f_2	0	1	0	0	2	4
f_3	0	0	1	2	0	4

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
x^2f_1	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
xyf_1	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
y^2f_1	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
xzf_1	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
yzf_1	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
z^2f_1	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
x^2f_2	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
xyf_2	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
y^2f_2	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
xzf_2	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
yzf_2	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
z^2f_2	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
x^2f_3	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
xyf_3	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
y^2f_3	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
xzf_3	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
yzf_3	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
z^2f_3	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

The F_5 algorithm

[Faugère, 2002]

Let $\mathbb{k} = \mathbb{F}_7$, $\succ =$ grevlex.

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$$f_3 = 4x^2 + xy + 4y^2 + 3xz + 5yz + 2z^2$$

$\widetilde{\mathcal{M}}_2$

	x^2	xy	y^2	xz	yz	z^2
f_1	1	1	2	1	1	4
f_2	0	1	0	0	2	4
f_3	0	0	1	2	0	4

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
x^2f_1	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
xyf_1	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
y^2f_1	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
xzf_1	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
yzf_1	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
z^2f_1	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
x^2f_2	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
xyf_2	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
y^2f_2	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
xzf_2	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
yzf_2	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
z^2f_2	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
x^2f_3	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
xyf_3	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
y^2f_3	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
xzf_3	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
yzf_3	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
z^2f_3	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

The F_5 algorithm [Faugère, 2002]

Let $\mathbb{k} = \mathbb{F}_7$, $\succ =$ grevlex.

$$f_1 = 5x^2 + 5xy + 3y^2 + 5xz + 5yz + 6z^2$$
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$$f_3 = 4x^2 + xy + 4y^2 + 3xz + 5yz + 2z^2$$

$\widetilde{\mathcal{M}}_2$

	x^2	xy	y^2	xz	yz	z^2
f_1	1	1	2	1	1	4
f_2	0	1	0	0	2	4
f_3	0	0	1	2	0	4

Lazard: $\mathcal{M}_4$ is 18×15.

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
x^2f_1	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
xyf_1	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
y^2f_1	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
xzf_1	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
yzf_1	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
z^2f_1	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
x^2f_2	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
xyf_2	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
y^2f_2	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
xzf_2	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
yzf_2	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
z^2f_2	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
x^2f_3	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
xyf_3	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
y^2f_3	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
xzf_3	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
yzf_3	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
z^2f_3	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

The F_5 algorithm

[Faugère, 2002]

Let $\mathbb{k} = \mathbb{F}_7$, $\succ =$ grevlex.

$$f_1 = 5x^2 + 5xy + 3y^2 + 5xz + 5yz + 6z^2$$
$$f_2 = 2x^2 + xy + 4y^2 + 2xz + 4z^2$$
$$f_3 = 4x^2 + xy + 4y^2 + 3xz + 5yz + 2z^2$$

$\widetilde{\mathcal{M}}_2$

	x^2	xy	y^2	xz	yz	z^2
f_1	1	1	2	1	1	4
f_2	0	1	0	0	2	4
f_3	0	0	1	2	0	4

Lazard: $\mathcal{M}_4$ is 18×15.

F_5: $\mathcal{M}_4$ is 15×15 and full rank!

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
x^2f_1	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
xyf_1	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
y^2f_1	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
xzf_1	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
yzf_1	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
z^2f_1	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
x^2f_2	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
xyf_2	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
y^2f_2	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
xzf_2	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
yzf_2	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
z^2f_2	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
x^2f_3	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
xyf_3	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
y^2f_3	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
xzf_3	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
yzf_3	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
z^2f_3	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

The F_5 algorithm

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Let $\mathbb{k} = \mathbb{F}_7$, $\succ = \text{grevlex}$.

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$$f_3 = 4x^2 + xy + 4y^2 + 3xz + 5yz + 2z^2$$

$\widetilde{\mathcal{M}}_2$

	x^2	xy	y^2	xz	yz	z^2
f_1	1	1	2	1	1	4
f_2	0	1	0	0	2	4
f_3	0	0	1	2	0	4

Lazard: $\mathcal{M}_4$ is 18×15.

F_5: $\mathcal{M}_4$ is 15×15 and full rank!

zero rows in $\widetilde{\mathcal{M}}_4$
$\downarrow$
zero rows in $\widetilde{\mathcal{M}}_5$

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
x^2f_1	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
xyf_1	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
y^2f_1	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
xzf_1	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
yzf_1	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
z^2f_1	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
x^2f_2	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
xyf_2	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
y^2f_2	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
xzf_2	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
yzf_2	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
z^2f_2	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
x^2f_3	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
xyf_3	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
y^2f_3	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
xzf_3	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
yzf_3	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
z^2f_3	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

The F_5 algorithm

[Faugère, 2002]

Let $\mathbb{k} = \mathbb{F}_7$, $\succ =$ grevlex.

$$f_1 = 5x^2 + 5xy + 3y^2 + 5xz + 5yz + 6z^2$$
$$f_2 = 2x^2 + xy + 4y^2 + 2xz + 4z^2$$
$$f_3 = 4x^2 + xy + 4y^2 + 3xz + 5yz + 2z^2$$

$(f_1, \ldots, f_m)$ generic

$\Downarrow$

$\{$

$\widetilde{\mathcal{M}}_2$

	x^2	xy	y^2	xz	yz	z^2
f_1	1	1	2	1	1	4
f_2	0	1	0	0	2	4
f_3	0	0	1	2	0	4

Lazard: $\mathcal{M}_4$ is 18×15.

F_5: $\mathcal{M}_4$ is 15×15 and full rank!

zero rows in $\widetilde{\mathcal{M}}_4$

$\downarrow$

zero rows in $\widetilde{\mathcal{M}}_5$

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
x^2f_1	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
xyf_1	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
y^2f_1	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
xzf_1	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
yzf_1	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
z^2f_1	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
x^2f_2	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
xyf_2	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
y^2f_2	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
xzf_2	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
yzf_2	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
z^2f_2	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
x^2f_3	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
xyf_3	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
y^2f_3	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
xzf_3	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
yzf_3	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
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The F_5 algorithm

[Faugère, 2002]

Let $\mathbb{k} = \mathbb{F}_7$, $\succ =$ grevlex.

$(f_1, \ldots, f_m)$ generic

$\Downarrow$

{ **No** reductions to zero.

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$\widetilde{\mathcal{M}}_2$

	x^2	xy	y^2	xz	yz	z^2
f_1	1	1	2	1	1	4
f_2	0	1	0	0	2	4
f_3	0	0	1	2	0	4

Lazard: $\mathcal{M}_4$ is 18×15.

F_5: $\mathcal{M}_4$ is 15×15 and full rank!

zero rows in $\widetilde{\mathcal{M}}_4$

$\downarrow$

zero rows in $\widetilde{\mathcal{M}}_5$

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
x^2f_1	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
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yzf_1	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
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x^2f_2	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
xyf_2	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
y^2f_2	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
xzf_2	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
yzf_2	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
z^2f_2	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
x^2f_3	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
xyf_3	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
y^2f_3	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
xzf_3	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
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$\Downarrow$
- **No** reductions to zero.
- Precise complexity analysis [Bardet, Faugère, Salvy 2015]

$\widetilde{\mathcal{M}}_2$

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	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
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y^2f_2	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
xzf_2	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
yzf_2	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
z^2f_2	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
x^2f_3	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
xyf_3	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
y^2f_3	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
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										x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
										6	0	0	0	0	0
										0	6	0	0	0	0
y^2f_1	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
xzf_1	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
yzf_1	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
z^2f_1	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
x^2f_2	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
xyf_2	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
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xzf_2	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
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z^2f_2	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
x^2f_3	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
xyf_3	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
y^2f_3	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
xzf_3	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
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Determinantal systems are not generic!

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										0	5	5	0	6	0
										5	5	3	5	5	6
										4	0	0	0	0	0
										0	4	0	0	0	0
										0	0	4	0	0	0
xzf_2	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
yzf_2	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
z^2f_2	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
x^2f_3	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
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y^2f_3	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
xzf_3	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
yzf_3	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
z^2f_3	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

Determinantal systems are not generic!

How do we remove reductions to zero?

Contributions

[G., Neiger, Safey El Din, 2024]

Problem

Given $F = (f_1, \ldots, f_p) \subseteq \mathbb{k}[x_1, \ldots, x_n]$ and $g \in \mathbb{k}[x_1, \ldots, x_n]$, all homogeneous of degree δ, compute a grevlex Gröbner basis of

$$\mathcal{I}(g, F) = \langle F \rangle + \mathcal{I}_{p+1}(\mathsf{jac}(g, F)).$$

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New F_5-type criterion

Leading monomials of columns of $\mathrm{jac}(g, F)$

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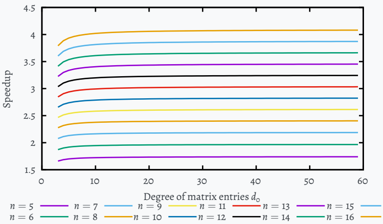

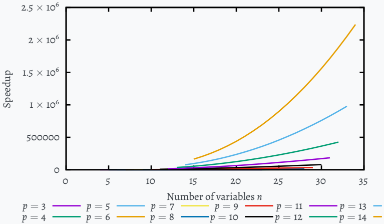

Syzygies

Definition (Syzygy)

$(a_1, \ldots, a_m) \in \mathbb{k}[x_1, \ldots, x_n]^m$ with

$$a_1 f_1 + \cdots + a_m f_m = 0$$

is called a **syzygy** of $f_1, \ldots, f_m$.

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$$\Big\Downarrow f_i f_j = f_j f_i$$

$$\underbrace{\mathsf{LT}(f_i) f_j}_{\text{row of Macaulay matrix}} = \underbrace{f_j f_i - \mathsf{tail}(f_i) f_j}_{\text{combination of rows of Macaulay matrix}}$$

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Syzygies of F ⟷ Reductions to zero in F_5

Theorem (Hilbert's syzygy theorem) [Hilbert, 1890]

Free resolution $0 \to \mathcal{E}_\ell \xrightarrow{d_\ell} \mathcal{E}_{\ell-1} \xrightarrow{d_{\ell-1}} \cdots \to \mathcal{E}_1 \xrightarrow{d_1} \mathcal{E}_0 \xrightarrow{\epsilon} \langle F \rangle \to 0 \Longrightarrow$

$$\mathrm{Syz}_k(F) = \ker(d_k) = \mathrm{im}(d_{k+1}).$$

The Eagon-Northcott complex

$$\begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} & f_{15} \\ f_{21} & f_{22} & f_{23} & f_{24} & f_{25} \\ f_{31} & f_{32} & f_{33} & f_{34} & f_{35} \end{pmatrix}$$

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$$f_{12} \begin{vmatrix} f_{13} & f_{14} & f_{15} \\ f_{23} & f_{24} & f_{25} \\ f_{33} & f_{34} & f_{35} \end{vmatrix} - f_{13} \begin{vmatrix} f_{12} & f_{14} & f_{15} \\ f_{22} & f_{24} & f_{25} \\ f_{32} & f_{34} & f_{35} \end{vmatrix} + f_{14} \begin{vmatrix} f_{12} & f_{13} & f_{15} \\ f_{22} & f_{23} & f_{25} \\ f_{32} & f_{33} & f_{35} \end{vmatrix} - f_{15} \begin{vmatrix} f_{12} & f_{13} & f_{14} \\ f_{22} & f_{23} & f_{24} \\ f_{32} & f_{33} & f_{34} \end{vmatrix}$$

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Theorem **[Eagon, Northcott, 1962]**

If M is a $p \times q$ matrix of generic polynomials, the syzygies of this form generate $\mathrm{Syz}(F_p(M))$.

$$\begin{vmatrix} f_{33} & f_{34} & f_{35} \end{vmatrix} \quad \begin{vmatrix} f_{32} & f_{34} & f_{35} \end{vmatrix} \quad \begin{vmatrix} f_{32} & f_{33} & f_{35} \end{vmatrix} \quad \begin{vmatrix} f_{32} & f_{33} & f_{34} \end{vmatrix}$$

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The algorithm

$$
\begin{aligned}
F &= (f_1, \ldots, f_p) \subseteq \mathbb{k}[x_1, \ldots, x_n] \\
g &\in \mathbb{k}[x_1, \ldots, x_n]
\end{aligned}
$$

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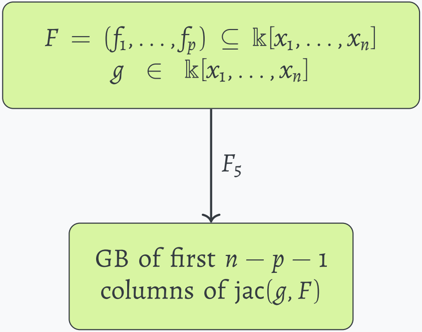

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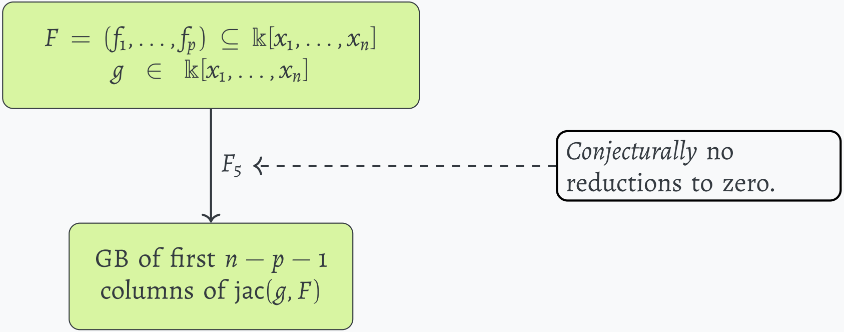

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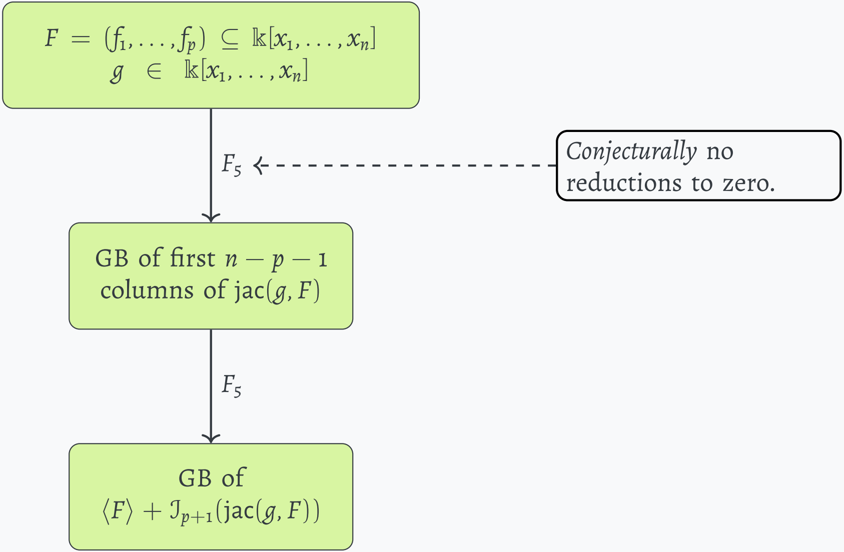

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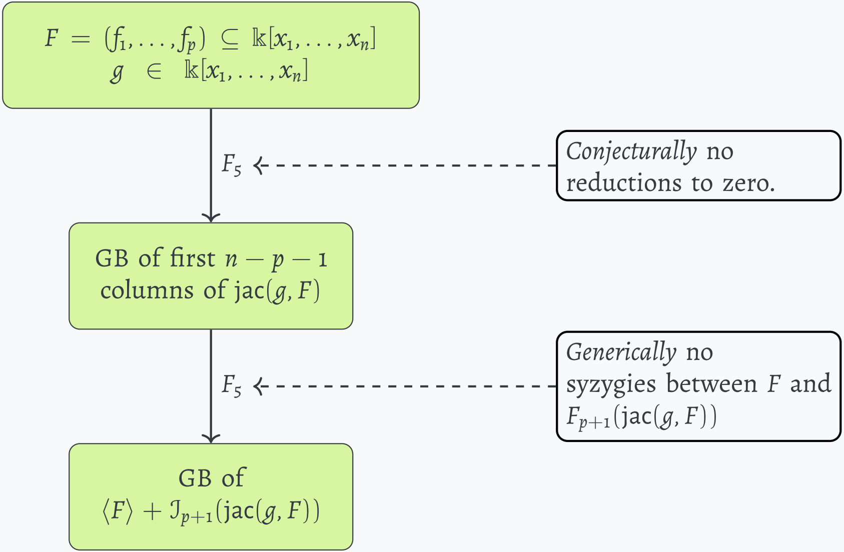

A complexity analysis

Theorem **[Spaenlehauer, 2013]**

$F \in \Bbbk[x_1, \dots, x_n]$ a regular sequence and $g \in \Bbbk[x_1, \dots, x_n]$ generic, then $\mathsf{Syz}(F \cup F_{p+1}(\mathsf{jac}(g, F))) = \mathsf{Syz}(F) \oplus \mathsf{Syz}(F_{p+1}(\mathsf{jac}(g, F)))$.

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Complexity

If $F = (f_1, \ldots, f_p) \subseteq \mathbb{k}[x_1, \ldots, x_n]$, $g \in \mathbb{k}[x_1, \ldots, x_n]$ all homogeneous of degree δ, and $\mathsf{HF}^{(j)}(d)$ is the Hilbert function of the first j columns of $\mathsf{jac}(g, F)$,

$$\sum_{j=1}^{n-p-1} \mathsf{HF}^{(j)}(d - (p+1)\delta) \binom{n-j-1}{p}$$

rows removed from $\mathcal{M}_d(F \cup F_{p+1}(\mathsf{jac}(g, F)))$.

Experimental results

M a $p \times (n + p - 1)$ matrix of homogeneous polynomials of degree d_0 in n variables.

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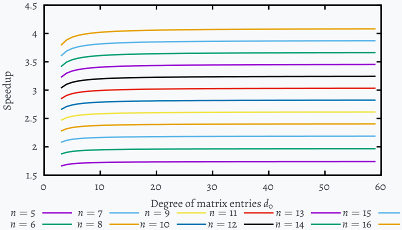
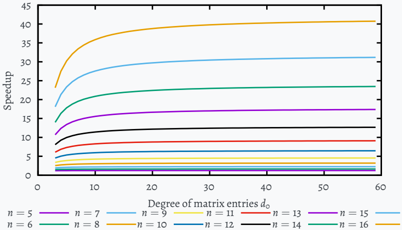


"Optimal" algorithm



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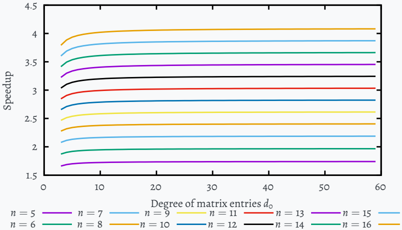

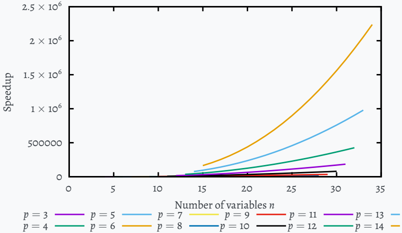

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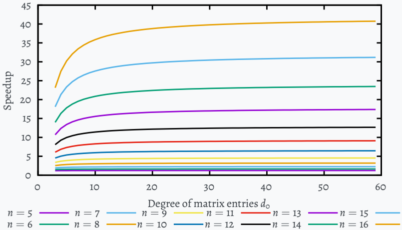

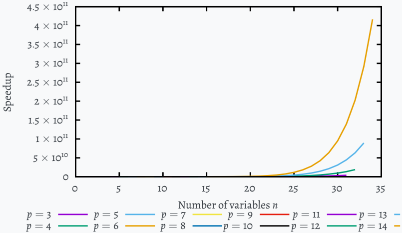

Conclusion and perspectives

Summary

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Thank you!