Gröbner bases of maximal minors and critical point computations

Journées nationales de calcul formel
March 6, 2024

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Critical points

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> **Problem**
>
> Compute the critical points of a polynomial g restricted to an algebraic set $V(F)$.
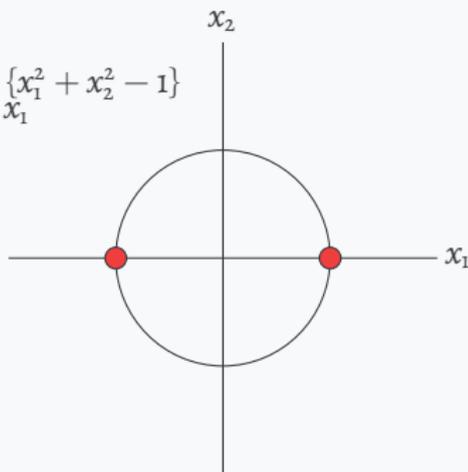
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Compute the critical points of a polynomial g restricted to an algebraic set $V(F)$.

Critical points of $g|_{V(F)}$ ← → Variety defined by F and the maximal minors of $\mathrm{jac}(g, F)$.

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$F = \{x_1^2 + x_2^2 - 1\}$
$g = x_1$
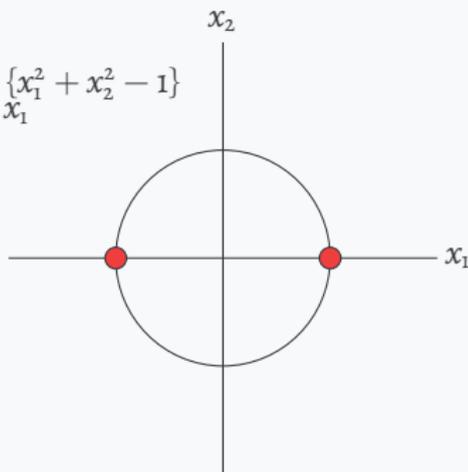
x_1

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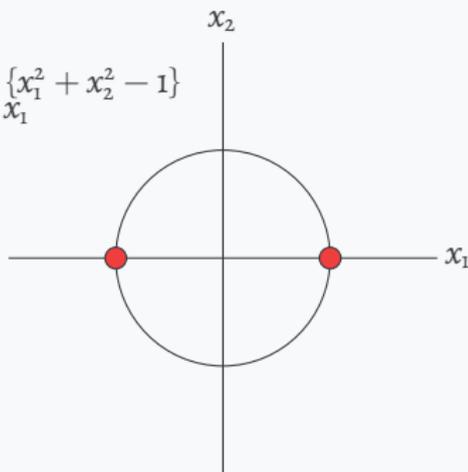$$g = x_1$$

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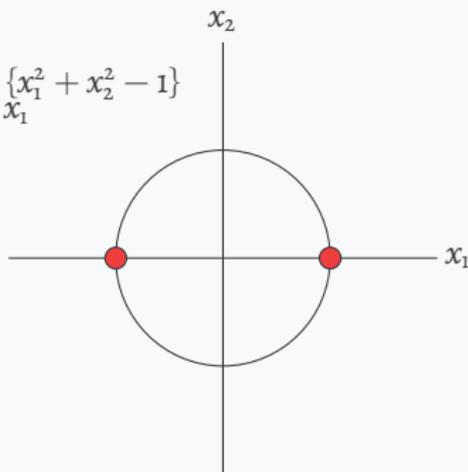
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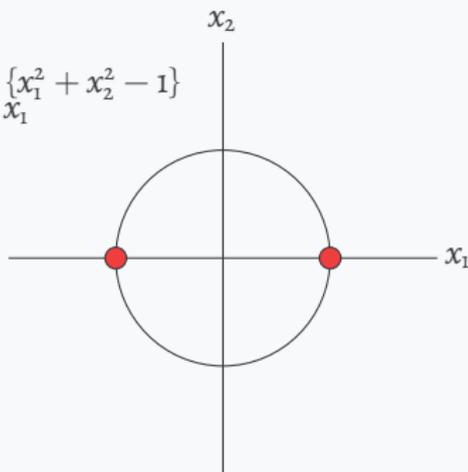\end{cases}
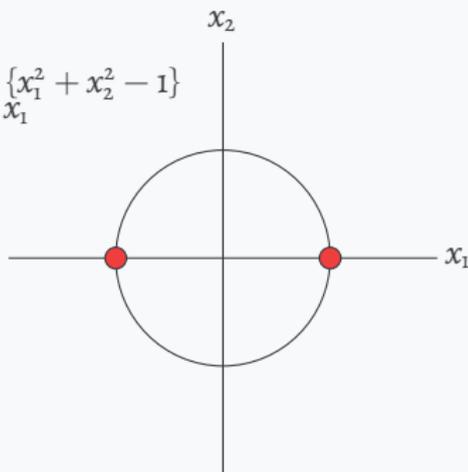$$

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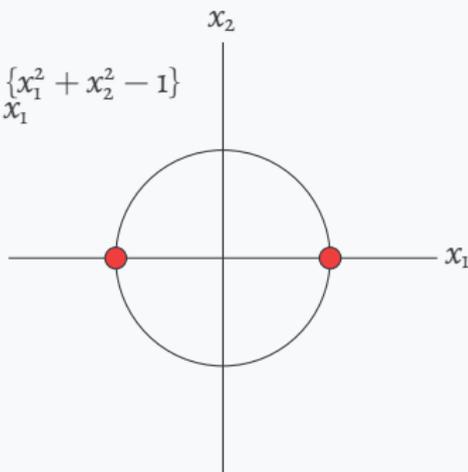
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Notation

$F_{p+1}(M) = $ system of $(p+1)$-minors of M

$$\mathcal{J}_{p+1}(M) = \langle F_{p+1}(M) \rangle$$

Gröbner bases

$F \subseteq \Bbbk[x_1, \ldots, x_n]$ a polynomial system.

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Definition (Gröbner bases)

A $\succ$-*Gröbner basis* is a finite generating set G for $\langle F \rangle$ such that

$$\langle \mathsf{LM}_\succ(G) \rangle = \mathsf{LM}_\succ(\langle F \rangle).$$

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$$\boxed{F} \xrightarrow[\text{grevlex order}]{\substack{\text{Compute} \\ \text{Gröbner basis}}} \boxed{\textbf{\textit{GB}}_{grevlex}} \xrightarrow[\text{FGLM}]{\substack{\text{Change} \\ \text{of ordering}}} \boxed{\textbf{\textit{GB}}_{lex}}$$

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F → **Compute Gröbner basis** / **grevlex order** → $GB_{grevlex}$ → **Change of ordering** / **FGLM** → GB_{lex}

Complexity

Doubly exponential in the number of variables. **[Mayr, Mayer, 1982]**

For zero-dimensional systems:

$$\max_{f \in F}\{\deg f\}^{O(\# \text{ of variables})} \qquad \textbf{[Lazard, 1983]}$$

Macaulay matrices - linearization

Assume F homogeneous.

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$$
\begin{cases}
\mathbf{f_1} = 2x^2 + 11xy - y^2 \\
\mathbf{f_2} = 4x^2 + xy - 2y^2 \\
\mathbf{f_3} = -6x^2 - xy + y^2
\end{cases}
$$

$$
\begin{array}{cccc}
x^3 & x^2y & xy^2 & y^3
\end{array}
$$

$$
\begin{pmatrix}
& & & \\
& & & \\
& & & \\
& & & \\
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$\xrightarrow{\cdot x}$

$$x\mathbf{f_1} \begin{array}{c} \begin{array}{cccc} x^3 & x^2y & xy^2 & y^3 \end{array} \\ \left(\begin{array}{cccc} 2 & 11 & -1 & 0 \\ & & & \\ & & & \\ & & & \\ & & & \\ \end{array} \right) \end{array}$$

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$\cdot x$

$\cdot y$

$x\mathbf{f_1}$

$y\mathbf{f_1}$

	x^3	x^2y	xy^2	y^3
$x\mathbf{f_1}$	2	11	-1	0
$y\mathbf{f_1}$	0	2	11	-1

3

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$$\begin{array}{c} \quad\quad\quad\quad\quad\quad x^3 \quad x^2y \quad xy^2 \quad y^3 \\ \begin{array}{c} x\mathbf{f_1} \\ y\mathbf{f_1} \\ x\mathbf{f_2} \end{array} \left(\begin{array}{cccc} 2 & 11 & -1 & 0 \\ 0 & 2 & 11 & -1 \\ 4 & 1 & -2 & 0 \end{array} \right. \end{array}$$

with maps $\cdot x$, $\cdot y$ giving $x\mathbf{f_1}$, $y\mathbf{f_1}$ and $\cdot x$ giving $x\mathbf{f_2}$.

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$$\begin{array}{c} \cdot x \\ \cdot y \\ \cdot x \\ \cdot y \end{array} \rightarrow \begin{array}{c} x\mathbf{f_1} \\ y\mathbf{f_1} \\ x\mathbf{f_2} \\ y\mathbf{f_2} \end{array}$$

$$\begin{array}{c} & x^3 & x^2y & xy^2 & y^3 \\ \begin{matrix} x\mathbf{f_1} \\ y\mathbf{f_1} \\ x\mathbf{f_2} \\ y\mathbf{f_2} \end{matrix} & \begin{pmatrix} 2 & 11 & -1 & 0 \\ 0 & 2 & 11 & -1 \\ 4 & 1 & -2 & 0 \\ 0 & 4 & 1 & -2 \end{pmatrix} \end{array}$$

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	x^3	x^2y	xy^2	y^3
$x\mathbf{f_1}$	2	11	−1	0
$y\mathbf{f_1}$	0	2	11	−1
$x\mathbf{f_2}$	4	1	−2	0
$y\mathbf{f_2}$	0	4	1	−2
$x\mathbf{f_3}$	−6	−1	1	0

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The rows of the echelonization of the Macaulay matrix of F in degree d form the elements of degree d of a $\succ$-Gröbner basis for F.

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With maps $\cdot x$, $\cdot y$, $\cdot x$, $\cdot y$, $\cdot y$, $\cdot x$, $\cdot y$.

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Theorem (Macaulay bound)   **[Lazard, 1983]**

The maximum degree of a polynomial in the **grevlex** Gröbner basis of a generic polynomial system $f_1, \ldots, f_m$ is

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Theorem **[Spaenlehauer, 2013]**

$F = (f_1, \ldots, f_p) \subseteq \Bbbk[x_1, \ldots, x_n]$, $g \in \Bbbk[x_1, \ldots, x_n]$ all homogeneous of degree δ. Generically, the number of arithmetic operations in $\Bbbk$ required to compute a grevlex GB of $\mathcal{I}(g, F) = \langle F \rangle + \mathcal{I}_{p+1}(\mathrm{jac}(g, F))$ is in

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> **Theorem (Maca...** **[Lazard, 1983]**
>
> The maximum ... ner basis of a generic polyno...
>
> **This upper bound is not very sharp since the Macaulay matrices are not full rank!**
>
> ⌐ algebraic property: regularity
>
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3

The F_5 algorithm

Let $\Bbbk = \mathbb{F}_7$, $\succ = \text{grevlex}$.

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$$\widetilde{\mathcal{M}}_2$$

	x^2	xy	y^2	xz	yz	z^2
f_1	1	1	2	1	1	4
f_2	0	1	0	0	2	4
f_3	0	0	1	2	0	4

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$$\widetilde{\mathcal{M}}_2 \qquad\qquad\qquad \mathcal{M}_4$$

	x^2	xy	y^2	xz	yz	z^2
f_1	1	1	2	1	1	4
f_2	0	1	0	0	2	4
f_3	0	0	1	2	0	4

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
x^2f_1	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
xyf_1	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
y^2f_1	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
xzf_1	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
yzf_1	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
z^2f_1	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
x^2f_2	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
xyf_2	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
y^2f_2	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
xzf_2	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
yzf_2	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
z^2f_2	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
x^2f_3	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
xyf_3	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
y^2f_3	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
xzf_3	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
yzf_3	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
z^2f_3	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

The F_5 algorithm

Let $\mathbb{k} = \mathbb{F}_7$, $\succ =$ grevlex .

$$f_1 = 5x^2 + 5xy + 3y^2 + 5xz + 5yz + 6z^2$$
$$f_2 = 2x^2 + xy + 4y^2 + 2xz + 4z^2$$
$$f_3 = 4x^2 + xy + 4y^2 + 3xz + 5yz + 2z^2$$

$\widetilde{\mathcal{M}}_2$

	x^2	xy	y^2	xz	yz	z^2
f_1	1	1	2	1	1	4
f_2	0	1	0	0	2	4
f_3	0	0	1	2	0	4

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
x^2f_1	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
xyf_1	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
y^2f_1	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
xzf_1	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
yzf_1	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
z^2f_1	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
x^2f_2	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
xyf_2	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
y^2f_2	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
xzf_2	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
yzf_2	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
z^2f_2	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
x^2f_3	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
xyf_3	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
y^2f_3	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
xzf_3	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
yzf_3	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
z^2f_3	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

The F_5 algorithm

Let $\Bbbk = \mathbb{F}_7$, $\succ = \text{grevlex}$.

$$f_1 = 5x^2 + 5xy + 3y^2 + 5xz + 5yz + 6z^2$$
$$f_2 = 2x^2 + xy + 4y^2 + 2xz + 4z^2$$
$$f_3 = 4x^2 + xy + 4y^2 + 3xz + 5yz + 2z^2$$

$\widetilde{\mathcal{M}}_2$

	x^2	xy	y^2	xz	yz	z^2
f_1	1	1	2	1	1	4
f_2	0	1	0	0	2	4
f_3	0	0	1	2	0	4

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
x^2f_1	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
xyf_1	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
y^2f_1	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
xzf_1	0	0	0	0	0	5	5	0	0	5	5	0	6	0	0
yzf_1	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
z^2f_1	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
x^2f_2	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
xyf_2	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
y^2f_2	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
xzf_2	0	0	0	0	0	2	1	4	0	0	2	0	4	0	0
yzf_2	0	0	0	0	0	0	2	1	4	0	0	2	0	4	0
z^2f_2	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
x^2f_3	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
xyf_3	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
y^2f_3	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
xzf_3	0	0	0	0	0	4	1	4	0	0	3	5	2	0	0
yzf_3	0	0	0	0	0	0	4	1	4	0	0	3	5	2	0
z^2f_3	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

The F_5 algorithm

Let $\Bbbk = \mathbb{F}_7$, $\succ = \text{grevlex}$.

$$f_1 = 5x^2 + 5xy + 3y^2 + 5xz + 5yz + 6z^2$$
$$f_2 = 2x^2 + xy + 4y^2 + 2xz + 4z^2$$
$$f_3 = 4x^2 + xy + 4y^2 + 3xz + 5yz + 2z^2$$

$\widetilde{\mathcal{M}}_2$

	x^2	xy	y^2	xz	yz	z^2
f_1	1	1	2	1	1	4
f_2	0	1	0	0	2	4
f_3	0	0	1	2	0	4

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
x^2f_1	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
xyf_1	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
y^2f_1	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
xzf_1	0	0	0	0	0	5	5	0	0	5	5	0	6	0	0
yzf_1	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
z^2f_1	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
x^2f_2	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
xyf_2	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
y^2f_2	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
xzf_2	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
yzf_2	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
z^2f_2	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
x^2f_3	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
xyf_3	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
y^2f_3	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
xzf_3	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
yzf_3	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
z^2f_3	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

The F_5 algorithm

Let $\Bbbk = \mathbb{F}_7$, $\succ = \text{grevlex}$.

$$f_1 = 5x^2 + 5xy + 3y^2 + 5xz + 5yz + 6z^2$$
$$f_2 = 2x^2 + xy + 4y^2 + 2xz + 4z^2$$
$$f_3 = 4x^2 + xy + 4y^2 + 3xz + 5yz + 2z^2$$

$\widetilde{\mathcal{M}}_2$

	x^2	xy	y^2	xz	yz	z^2
f_1	1	1	2	1	1	4
f_2	0	1	0	0	2	4
f_3	0	0	1	2	0	4

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
x^2f_1	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
xyf_1	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
y^2f_1	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
xzf_1	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
yzf_1	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
z^2f_1	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
x^2f_2	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
xyf_2	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
y^2f_2	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
xzf_2	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
yzf_2	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
z^2f_2	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
x^2f_3	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
xyf_3	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
y^2f_3	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
xzf_3	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
yzf_3	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
z^2f_3	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

The F_5 algorithm

[Faugère, 2002]

Let $\Bbbk = \mathbb{F}_7$, $\succ = \text{grevlex}$.

$$f_1 = 5x^2 + 5xy + 3y^2 + 5xz + 5yz + 6z^2$$
$$f_2 = 2x^2 + xy + 4y^2 + 2xz + 4z^2$$
$$f_3 = 4x^2 + xy + 4y^2 + 3xz + 5yz + 2z^2$$

$$\widetilde{\mathcal{M}}_2$$

	x^2	xy	y^2	xz	yz	z^2
f_1	1	1	2	1	1	4
f_2	0	1	0	0	2	4
f_3	0	0	1	2	0	4

$$\mathcal{M}_4$$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
x^2f_1	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
xyf_1	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
y^2f_1	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
xzf_1	0	0	0	0	0	5	5	0	0	5	5	0	6	0	0
yzf_1	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
z^2f_1	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
x^2f_2	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
xyf_2	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
y^2f_2	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
xzf_2	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
yzf_2	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
z^2f_2	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
x^2f_3	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
xyf_3	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
y^2f_3	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
xzf_3	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
yzf_3	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
z^2f_3	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

4

Let $\Bbbk = \mathbb{F}_7$, $\succ = \text{grevlex}$.

$$f_1 = 5x^2 + 5xy + 3y^2 + 5xz + 5yz + 6z^2$$
$$f_2 = 2x^2 + xy + 4y^2 + 2xz + 4z^2$$
$$f_3 = 4x^2 + xy + 4y^2 + 3xz + 5yz + 2z^2$$

$\widetilde{\mathcal{M}}_2$

	x^2	xy	y^2	xz	yz	z^2
f_1	1	1	2	1	1	4
f_2	0	1	0	0	2	4
f_3	0	0	1	2	0	4

Lazard: $\mathcal{M}_4$ is 18×15.

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
x^2f_1	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
xyf_1	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
y^2f_1	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
xzf_1	0	0	0	0	0	5	5	0	0	5	5	0	6	0	0
yzf_1	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
z^2f_1	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
x^2f_2	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
xyf_2	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
y^2f_2	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
xzf_2	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
yzf_2	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
z^2f_2	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
x^2f_3	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
xyf_3	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
y^2f_3	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
xzf_3	0	0	0	0	0	4	1	4	0	3	0	0	5	0	0
yzf_3	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
z^2f_3	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

Let $\Bbbk = \mathbb{F}_7$, $\succ = \text{grevlex}$.

$$f_1 = 5x^2 + 5xy + 3y^2 + 5xz + 5yz + 6z^2$$
$$f_2 = 2x^2 + xy + 4y^2 + 2xz + 4z^2$$
$$f_3 = 4x^2 + xy + 4y^2 + 3xz + 5yz + 2z^2$$

$$\widetilde{\mathcal{M}}_2 \qquad\qquad \mathcal{M}_4$$

$\widetilde{\mathcal{M}}_2$:

	x^2	xy	y^2	xz	yz	z^2
f_1	1	1	2	1	1	4
f_2	0	1	0	0	2	4
f_3	0	0	1	2	0	4

Lazard: $\mathcal{M}_4$ is 18×15.

$\mathbf{F_5}$: $\mathcal{M}_4$ is 15×15 and full rank!

$\mathcal{M}_4$:

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
x^2f_1	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
xyf_1	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
y^2f_1	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
xzf_1	0	0	0	0	0	5	5	0	0	5	5	0	6	0	0
yzf_1	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
z^2f_1	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
x^2f_2	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
xyf_2	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
y^2f_2	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
xzf_2	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
yzf_2	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
z^2f_2	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
x^2f_3	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
xyf_3	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
y^2f_3	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
xzf_3	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
yzf_3	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
z^2f_3	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

The F_5 algorithm

Let $\Bbbk = \mathbb{F}_7$, $\succ = \text{grevlex}$.

$$f_1 = 5x^2 + 5xy + 3y^2 + 5xz + 5yz + 6z^2$$
$$f_2 = 2x^2 + xy + 4y^2 + 2xz + 4z^2$$
$$f_3 = 4x^2 + xy + 4y^2 + 3xz + 5yz + 2z^2$$

$\widetilde{\mathcal{M}}_2$ 　　　　　　　　　 $\mathcal{M}_4$

	x^2	xy	y^2	xz	yz	z^2
f_1	1	1	2	1	1	4
f_2	0	1	0	0	2	4
f_3	0	0	1	2	0	4

Lazard: $\mathcal{M}_4$ is 18×15.

F_5: $\mathcal{M}_4$ is 15×15 and full rank!

> **zero rows in** $\widetilde{\mathcal{M}}_4$
> $\downarrow$
> **zero rows in** $\widetilde{\mathcal{M}}_5$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
x^2f_1	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
xyf_1	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
y^2f_1	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
xzf_1	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
yzf_1	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
z^2f_1	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
x^2f_2	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
xyf_2	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
y^2f_2	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
xzf_2	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
yzf_2	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
z^2f_2	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
x^2f_3	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
xyf_3	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
y^2f_3	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
xzf_3	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
yzf_3	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
z^2f_3	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

The F_5 algorithm

Let $\Bbbk = \mathbb{F}_7$, $\succ =$ grevlex .

$f_1 = 5x^2 + 5xy + 3y^2 + 5xz + 5yz + 6z^2$

$f_2 = 2x^2 + xy + 4y^2 + 2xz + 4z^2$

$f_3 = 4x^2 + xy + 4y^2 + 3xz + 5yz + 2z^2$

$(f_1, \ldots, f_m)$ generic $\Longrightarrow$ $\Big\{$

$\widetilde{\mathcal{M}}_2$

$\mathcal{M}_4$

	x^2	xy	y^2	xz	yz	z^2
f_1	1	1	2	1	1	4
f_2	0	1	0	0	2	4
f_3	0	0	1	2	0	4

Lazard: $\mathcal{M}_4$ is 18×15.

F_5: $\mathcal{M}_4$ is 15×15 and full rank!

zero rows in $\widetilde{\mathcal{M}}_4$
$\downarrow$
zero rows in $\widetilde{\mathcal{M}}_5$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
x^2f_1	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
xyf_1	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
y^2f_1	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
xzf_1	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
yzf_1	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
z^2f_1	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
x^2f_2	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
xyf_2	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
y^2f_2	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
xzf_2	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
yzf_2	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
z^2f_2	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
x^2f_3	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
xyf_3	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
y^2f_3	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
xzf_3	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
yzf_3	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
z^2f_3	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

The F_5 algorithm

Let $\Bbbk = \mathbb{F}_7$, $\succ = \text{grevlex}$.

$$f_1 = 5x^2 + 5xy + 3y^2 + 5xz + 5yz + 6z^2$$
$$f_2 = 2x^2 + xy + 4y^2 + 2xz + 4z^2$$
$$f_3 = 4x^2 + xy + 4y^2 + 3xz + 5yz + 2z^2$$

$(f_1, \ldots, f_m)$ generic

$\Downarrow$

$\{$ **No** reductions to zero.

$$\widetilde{\mathcal{M}}_2$$

	x^2	xy	y^2	xz	yz	z^2
f_1	1	1	2	1	1	4
f_2	0	1	0	0	2	4
f_3	0	0	1	2	0	4

Lazard: $\mathcal{M}_4$ is 18×15.

$\mathbf{F_5}$: $\mathcal{M}_4$ is 15×15 and full rank!

zero rows in $\widetilde{\mathcal{M}}_4$
$\downarrow$
zero rows in $\widetilde{\mathcal{M}}_5$

$$\mathcal{M}_4$$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
x^2f_1	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
xyf_1	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
y^2f_1	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
xzf_1	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
yzf_1	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
z^2f_1	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
x^2f_2	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
xyf_2	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
y^2f_2	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
xzf_2	0	0	0	0	0	2	1	4	0	0	2	0	4	0	0
yzf_2	0	0	0	0	0	0	2	1	4	0	0	2	0	4	0
z^2f_2	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
x^2f_3	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
xyf_3	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
y^2f_3	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
xzf_3	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
yzf_3	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
z^2f_3	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

The F_5 algorithm

Let $\mathbb{k} = \mathbb{F}_7$, $\succ = $ grevlex .

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$\widetilde{\mathcal{M}}_2$

$(f_1, \ldots, f_m)$ generic

$\Downarrow$

$\begin{cases} \textbf{No} \text{ reductions to zero.} \\ \text{Precise complexity analysis} \end{cases}$ [Bardet, Faugère, Salvy 2015]

$\mathcal{M}_4$

	x^2	xy	y^2	xz	yz	z^2
f_1	1	1	2	1	1	4
f_2	0	1	0	0	2	4
f_3	0	0	1	2	0	4

Lazard: $\mathcal{M}_4$ is 18 × 15.
$\mathbf{F_5}$: $\mathcal{M}_4$ is 15 × 15 and full rank!

zero rows in $\widetilde{\mathcal{M}}_4$
↓
zero rows in $\widetilde{\mathcal{M}}_5$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
x^2f_1	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
xyf_1	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
y^2f_1	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
xzf_1	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
yzf_1	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
z^2f_1	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
x^2f_2	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
xyf_2	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
y^2f_2	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
xzf_2	0	0	0	0	0	2	1	4	0	0	2	0	4	0	0
yzf_2	0	0	0	0	0	0	2	1	4	0	0	2	0	4	0
z^2f_2	0	0	0	0	0	0	0	0	0	2	1	4	2	4	4
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xyf_3	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
y^2f_3	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
xzf_3	0	0	0	0	0	4	1	4	0	0	3	5	2	0	0
yzf_3	0	0	0	0	0	0	4	1	4	0	0	3	5	2	0
z^2f_3	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

The F_5 algorithm

Let $\Bbbk = \mathbb{F}_7$, $\succ = $ grevlex .

$f_1 = 5x^2 + 5xy + 3y^2 + 5xz + 5yz + 6z^2$

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$f_3 = 4x^2 + xy + 4$

$(f_1, \ldots, f_m)$ generic

‖

[Bardet, Faugère, Salvy] 2015

Determinantal systems are not generic!

$\widetilde{\mathcal{M}}_2$

	x^2	xy	y^2	x		
f_1	1	1	2			
f_2	0	1	0	0	2	4
f_3	0	0	1	2	0	4

Lazard: $\mathcal{M}_4$ is 18 × 15.

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zero rows in $\widetilde{\mathcal{M}}_4$

↓

zero rows in $\widetilde{\mathcal{M}}_5$

(large matrix $\widetilde{\mathcal{M}}_4$ with rows $y^2 f_1$, $xz f_1$, $yz f_1$, $z^2 f_1$, $x^2 f_2$, $xy f_2$, $y^2 f_2$, $xz f_2$, $yz f_2$, $z^2 f_2$, $x^2 f_3$, $xy f_3$, $y^2 f_3$, $xz f_3$, $yz f_3$, $z^2 f_3$ and columns including x^2z^2, xyz^2, y^2z^2, xz^3, yz^3, z^4)

Let $\Bbbk = \mathbb{F}_7$, $\succ = \text{grevlex}$.

$(f_1, \ldots, f_m)$ generic

$f_1 = 5x^2 + 5xy + 3y^2 + 5xz + 5yz + 6z^2$

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$f_3 = 4x^2 + xy + 4$

Determinantal systems are not generic!

[Bardet, Faugère, Salvy]
2015

$\widetilde{\mathcal{M}}_2$

How do we remove reductions to zero?

Lazard: $\mathcal{M}_4$ is

$\mathbf{F}_5$: $\mathcal{M}_4$ is 15×15 and full rank!

zero rows in $\widetilde{\mathcal{M}}_4$
$\downarrow$
zero rows in $\widetilde{\mathcal{M}}_5$

4

Contributions

> **Problem**
>
> Given $F = (f_1, \ldots, f_p) \subseteq \Bbbk[x_1, \ldots, x_n]$ and $g \in \Bbbk[x_1, \ldots, x_n]$, all homogeneous of degree δ, compute a grevlex Gröbner basis of
>
> $$\mathfrak{I}(g, F) = \langle F \rangle + \mathfrak{I}_{p+1}(\mathrm{jac}(g, F)).$$

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New F_5-type criterion

Leading monomials
of columns of $\mathrm{jac}(g, F)$

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Leading monomials of columns of $\mathrm{jac}(g, F)$ $\longrightarrow$ Reductions to zero when computing GB of $\mathcal{I}(g, F)$

Left plot: x-axis "Degree of matrix entries $d_\circ$", y-axis "Speedup". Legend: $n = 5$, $n = 6$, $n = 7$, $n = 8$, $n = 9$, $n = 10$, $n = 11$, $n = 12$, $n = 13$, $n = 14$, $n = 15$, $n = 16$.

Right plot: x-axis "Number of variables n", y-axis "Speedup". Legend: $p = 3$, $p = 4$, $p = 5$, $p = 6$, $p = 7$, $p = 8$, $p = 9$, $p = 10$, $p = 11$, $p = 12$, $p = 13$, $p = 14$.

Syzygies

Definition (Syzygy)

$(a_1, \ldots, a_m) \quad \in \quad \Bbbk[x_1, \ldots, x_n]^m$
with

$$a_1 f_1 + \cdots + a_m f_m = 0$$

is called a **syzygy** of $f_1, \ldots, f_m$.

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$$\Big\Downarrow f_i f_j = f_j f_i$$

$$\underbrace{\mathsf{LT}(f_i) f_j}_{\substack{\text{row of} \\ \text{Macaulay} \\ \text{matrix}}} = \underbrace{f_j f_i - \mathsf{tail}(f_i) f_j}_{\substack{\text{combination of} \\ \text{rows of} \\ \text{Macaulay matrix}}}$$

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$$\boxed{\text{Syzygies of } F} \longleftrightarrow \boxed{\begin{array}{c} \text{Reductions} \\ \text{to zero in } F_5 \end{array}}$$

Syzygies

Definition (Syzygy)

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with

$$a_1 f_1 + \cdots + a_m f_m = 0$$

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Syzygies of F $\longleftrightarrow$ Reductions to zero in F_5

Theorem (Hilbert's syzygy theorem) [Hilbert, 1890]

Free resolution $0 \to \mathcal{E}_\ell \xrightarrow{d_\ell} \mathcal{E}_{\ell-1} \xrightarrow{d_{\ell-1}} \cdots \to \mathcal{E}_1 \xrightarrow{d_1} \mathcal{E}_0 \xrightarrow{\epsilon} \langle F \rangle \to 0 \implies$

$$\mathsf{Syz}_k(F) = \ker(d_k) = \mathsf{im}(d_{k+1}).$$

The Eagon-Northcott complex

$$\begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} & f_{15} \\ f_{21} & f_{22} & f_{23} & f_{24} & f_{25} \\ f_{31} & f_{32} & f_{33} & f_{34} & f_{35} \end{pmatrix}$$

The Eagon-Northcott complex

$$\begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} & f_{15} \\ f_{21} & f_{22} & f_{23} & f_{24} & f_{25} \\ f_{31} & f_{32} & f_{33} & f_{34} & f_{35} \end{pmatrix} \longrightarrow \begin{pmatrix} f_{12} & f_{13} & f_{14} & f_{15} \\ f_{22} & f_{23} & f_{24} & f_{25} \\ f_{32} & f_{33} & f_{34} & f_{35} \end{pmatrix}$$

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The Eagon-Northcott complex

$$
\left(\begin{array}{ccccc}
f_{11} & f_{12} & f_{13} & f_{14} & f_{15} \\
f_{21} & f_{22} & f_{23} & f_{24} & f_{25} \\
f_{31} & f_{32} & f_{33} & f_{34} & f_{35}
\end{array}\right)
\longrightarrow
\left(\begin{array}{ccccc}
f_{12} & f_{13} & f_{14} & f_{15} \\
f_{12} & f_{13} & f_{14} & f_{15} \\
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\end{array}\right)
$$

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$$\begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} & f_{15} \\ f_{21} & f_{22} & f_{23} & f_{24} & f_{25} \\ f_{31} & f_{32} & f_{33} & f_{34} & f_{35} \end{pmatrix} \longrightarrow \begin{pmatrix} f_{12} & f_{13} & f_{14} & f_{15} \\ f_{12} & f_{13} & f_{14} & f_{15} \\ f_{22} & f_{23} & f_{24} & f_{25} \\ f_{32} & f_{33} & f_{34} & f_{35} \end{pmatrix}$$

$$f_{12}\begin{vmatrix} f_{13} & f_{14} & f_{15} \\ f_{23} & f_{24} & f_{25} \\ f_{33} & f_{34} & f_{35} \end{vmatrix} - f_{13}\begin{vmatrix} f_{12} & f_{14} & f_{15} \\ f_{22} & f_{24} & f_{25} \\ f_{32} & f_{34} & f_{35} \end{vmatrix} + f_{14}\begin{vmatrix} f_{12} & f_{13} & f_{15} \\ f_{22} & f_{23} & f_{25} \\ f_{32} & f_{33} & f_{35} \end{vmatrix} - f_{15}\begin{vmatrix} f_{12} & f_{13} & f_{14} \\ f_{22} & f_{23} & f_{24} \\ f_{32} & f_{33} & f_{34} \end{vmatrix}$$

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$$f_{22}\begin{vmatrix} f_{13} & f_{14} & f_{15} \\ f_{23} & f_{24} & f_{25} \\ f_{33} & f_{34} & f_{35} \end{vmatrix} - f_{23}\begin{vmatrix} f_{12} & f_{14} & f_{15} \\ f_{22} & f_{24} & f_{25} \\ f_{32} & f_{34} & f_{35} \end{vmatrix} + f_{24}\begin{vmatrix} f_{12} & f_{13} & f_{15} \\ f_{22} & f_{23} & f_{25} \\ f_{32} & f_{33} & f_{35} \end{vmatrix} - f_{25}\begin{vmatrix} f_{12} & f_{13} & f_{14} \\ f_{22} & f_{23} & f_{24} \\ f_{32} & f_{33} & f_{34} \end{vmatrix}$$

The Eagon-Northcott complex

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$$f_{22}\begin{vmatrix} f_{13} & f_{14} & f_{15} \\ f_{23} & f_{24} & f_{25} \\ f_{33} & f_{34} & f_{35} \end{vmatrix} - f_{23}\begin{vmatrix} f_{12} & f_{14} & f_{15} \\ f_{22} & f_{24} & f_{25} \\ f_{32} & f_{34} & f_{35} \end{vmatrix} + f_{24}\begin{vmatrix} f_{12} & f_{13} & f_{15} \\ f_{22} & f_{23} & f_{25} \\ f_{32} & f_{33} & f_{35} \end{vmatrix} - f_{25}\begin{vmatrix} f_{12} & f_{13} & f_{14} \\ f_{22} & f_{23} & f_{24} \\ f_{32} & f_{33} & f_{34} \end{vmatrix}$$

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$$\tau \in \mathsf{LM}_{\succ}(\langle f_{12}, f_{22}, f_{32}\rangle) \implies \tau\begin{vmatrix} f_{13} & f_{14} & f_{15} \\ f_{23} & f_{24} & f_{25} \\ f_{33} & f_{34} & f_{35} \end{vmatrix} \text{ reduces to zero.}$$

The Eagon-Northcott complex

$$
\begin{pmatrix}
f_{11} & f_{12} & f_{13} & f_{14} & f_{15} \\
f_{21} & f_{22} & f_{23} & f_{24} & f_{25} \\
f_{31} & f_{32} & f_{33} & f_{34} & f_{35}
\end{pmatrix}
\longrightarrow
\begin{pmatrix}
f_{12} & f_{13} & f_{14} & f_{15} \\
f_{22} & f_{23} & f_{24} & f_{25} \\
f_{32} & f_{33} & f_{34} & f_{35}
\end{pmatrix}
$$

Theorem [Eagon, Northcott, 1962]

If M is a $p \times q$ matrix of generic polynomials, the syzygies of this form generate $\mathsf{Syz}(F_p(M))$.

$$
J_{22}
\begin{vmatrix} J_{23} & J_{24} & J_{25} \\ f_{33} & f_{34} & f_{35} \end{vmatrix}
- J_{23}
\begin{vmatrix} J_{22} & J_{24} & J_{25} \\ f_{32} & f_{34} & f_{35} \end{vmatrix}
+ J_{24}
\begin{vmatrix} J_{22} & J_{23} & J_{25} \\ f_{32} & f_{33} & f_{35} \end{vmatrix}
- J_{25}
\begin{vmatrix} J_{22} & J_{23} & J_{24} \\ f_{32} & f_{33} & f_{34} \end{vmatrix}
$$

$$
f_{32}
\begin{vmatrix} f_{13} & f_{14} & f_{15} \\ f_{23} & f_{24} & f_{25} \\ f_{33} & f_{34} & f_{35} \end{vmatrix}
- f_{33}
\begin{vmatrix} f_{12} & f_{14} & f_{15} \\ f_{22} & f_{24} & f_{25} \\ f_{32} & f_{34} & f_{35} \end{vmatrix}
+ f_{34}
\begin{vmatrix} f_{12} & f_{13} & f_{15} \\ f_{22} & f_{23} & f_{25} \\ f_{32} & f_{33} & f_{35} \end{vmatrix}
- f_{35}
\begin{vmatrix} f_{12} & f_{13} & f_{14} \\ f_{22} & f_{23} & f_{24} \\ f_{32} & f_{33} & f_{34} \end{vmatrix}
$$

$$
\tau \in \mathsf{LM}_{\succ}(\langle f_{12}, f_{22}, f_{32} \rangle) \implies \tau
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$$

The algorithm

$$F = (f_1, \ldots, f_p) \subseteq \Bbbk[x_1, \ldots, x_n]$$
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F_5

GB of first $n - p - 1$
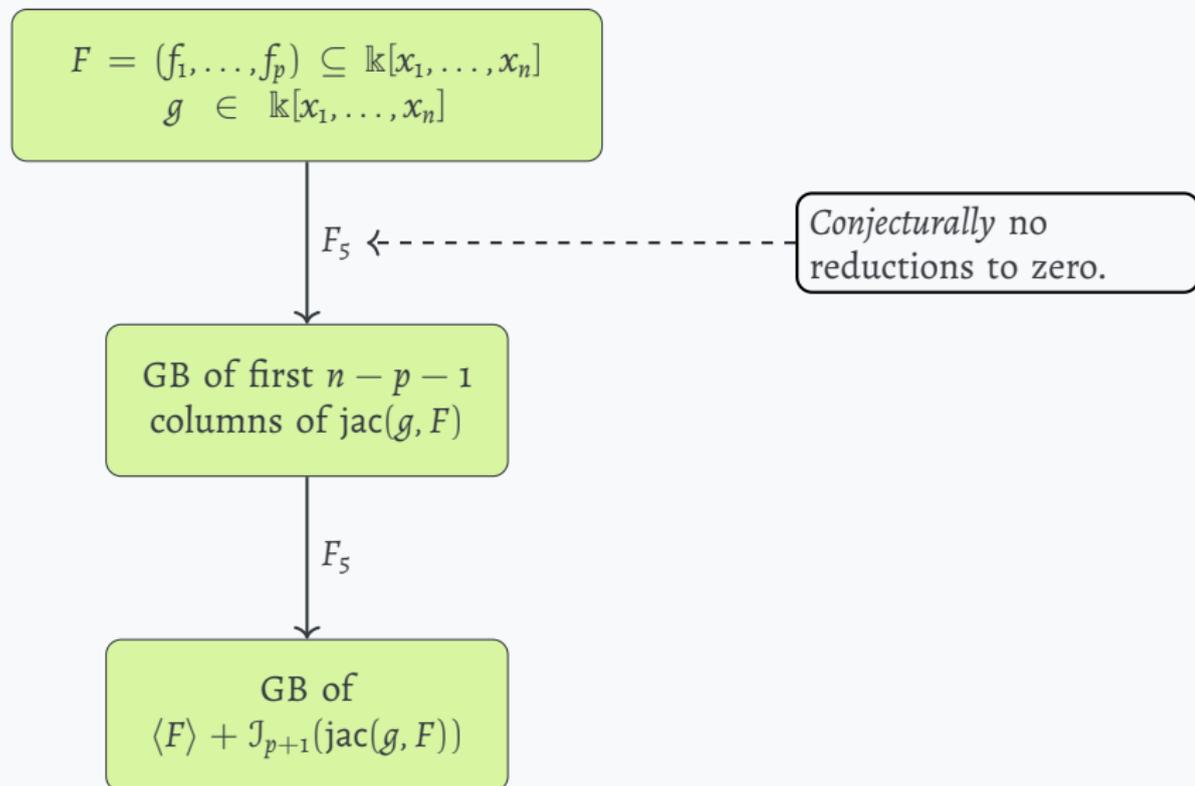columns of $\mathrm{jac}(g, F)$

The algorithm

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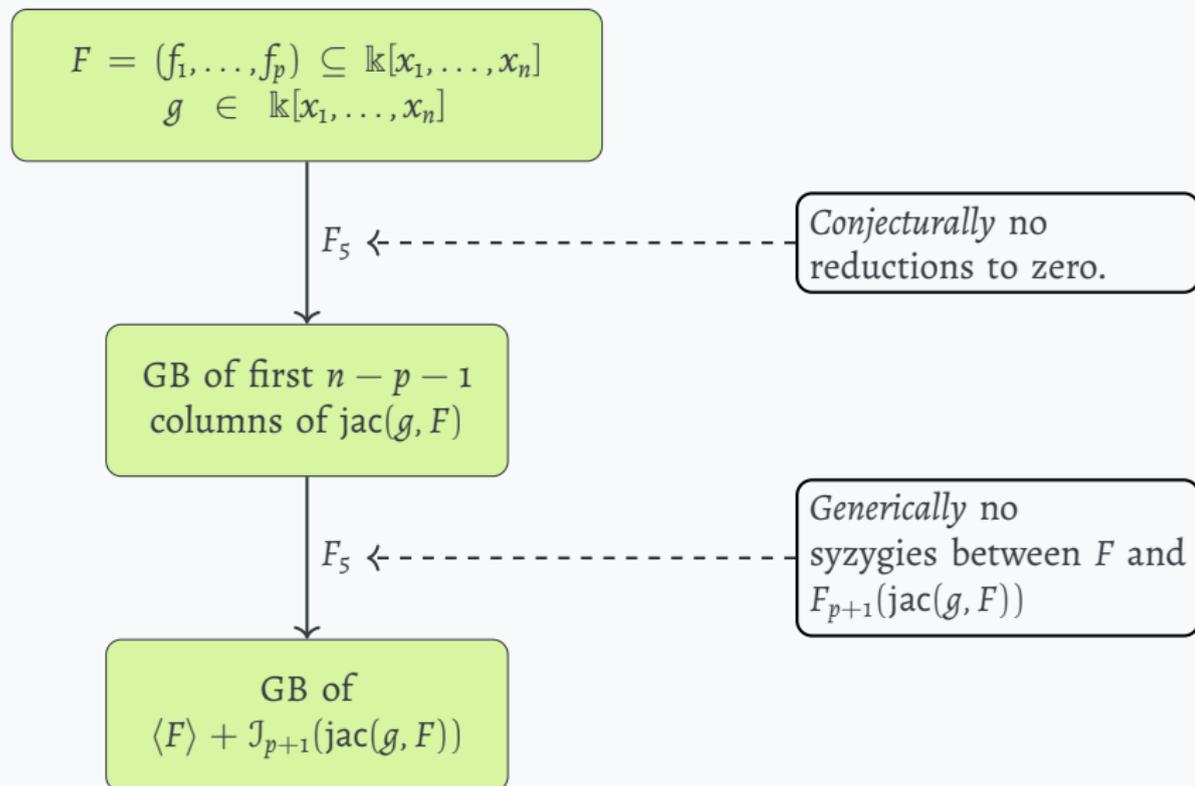$$g \in \Bbbk[x_1, \ldots, x_n]$$

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Conjecturally no reductions to zero.

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GB of $\langle F \rangle + \mathcal{I}_{p+1}(\mathrm{jac}(g, F))$

The algorithm

$$F = (f_1, \ldots, f_p) \subseteq \Bbbk[x_1, \ldots, x_n]$$
$$g \in \Bbbk[x_1, \ldots, x_n]$$

F_5 ← *Conjecturally* no reductions to zero.

GB of first $n - p - 1$ columns of $\mathrm{jac}(g, F)$

F_5 ← *Generically* no syzygies between F and $F_{p+1}(\mathrm{jac}(g, F))$

GB of $\langle F \rangle + \mathcal{I}_{p+1}(\mathrm{jac}(g, F))$

A complexity analysis

Theorem **[Spaenlehauer, 2013]**

$F \in \Bbbk[x_1, \ldots, x_n]$ a regular sequence and $g \in \Bbbk[x_1, \ldots, x_n]$ generic, then
$\mathsf{Syz}(F \cup F_{p+1}(\mathrm{jac}(g, F))) = \mathsf{Syz}(F) \oplus \mathsf{Syz}(F_{p+1}(\mathrm{jac}(g, F)))$.

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Eagon-Northcott complex $\longrightarrow$ Hilbert function of maximal minors

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Complexity

If $F = (f_1, \ldots, f_p) \subseteq \Bbbk[x_1, \ldots, x_n], g \in \Bbbk[x_1, \ldots, x_n]$ all homogeneous of degree δ, and $\mathsf{HF}^{(j)}(d)$ is the Hilbert function of the first j columns of $\mathrm{jac}(g, F)$,

$$\sum_{j=1}^{n-p-1} \mathsf{HF}^{(j)}(d - (p+1)\delta) \binom{n-j-1}{p}$$

rows removed from $\mathcal{M}_d(F \cup F_{p+1}(\mathrm{jac}(g, F)))$.

Experimental results

M a $p \times (n + p - 1)$ matrix of homogeneous polynomials of degree d_0 in n variables.
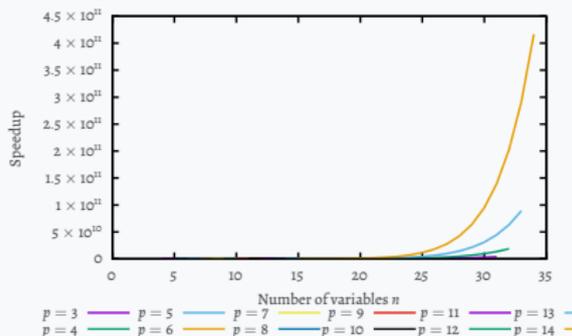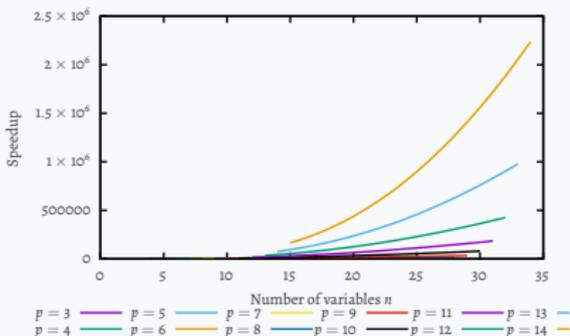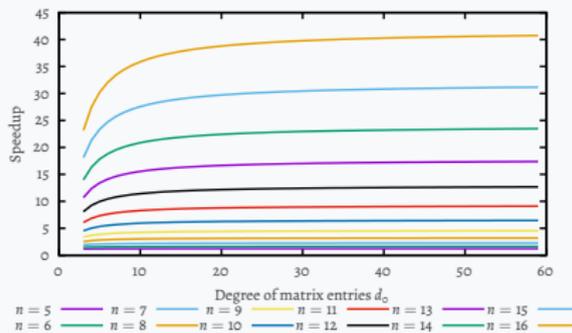
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Our algorithm
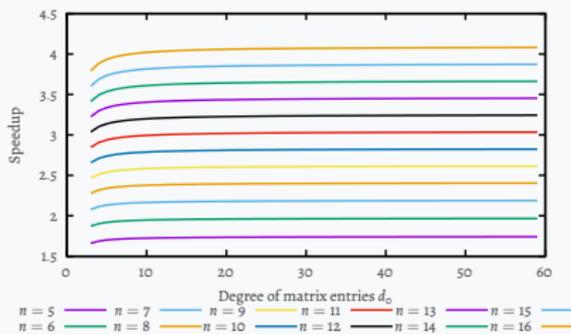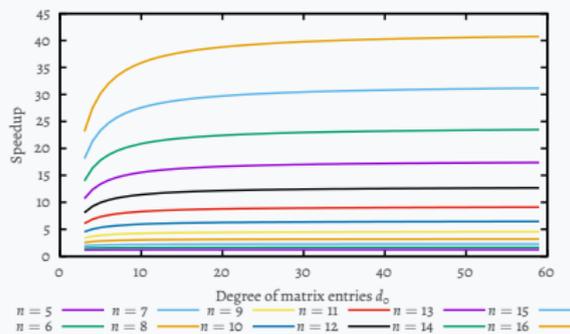
"Optimal" algorithm

Experimental results

M a $p \times (n + p - 1)$ matrix of homogeneous polynomials of degree d_o in n variables.
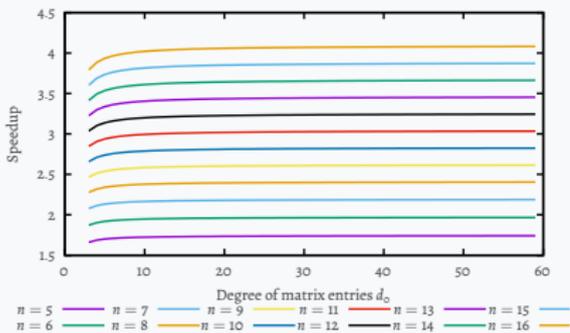
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THANK YOU!