Complexity aspects of Gröbner basis attacks on multivariate post-quantum cryptosystems

Young Cryptographers in Genova 2023

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Determinantal systems

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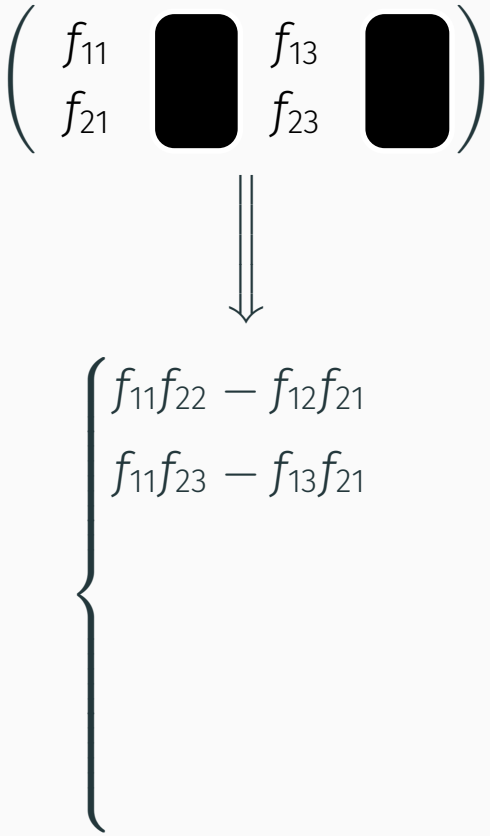

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$f_{i,j}$ are **linear forms** in $\mathbb{k}[x_1, \ldots, x_k]$.
Find $\boldsymbol{a} \in \overline{\mathbb{k}}^k$ with $\mathbf{rank}(M(\boldsymbol{a})) \leq r$.

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Security of post-quantum cryptosystems $\longleftrightarrow$ **Complexity** of MinRank

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$F \subseteq \mathbb{k}[x_1, \ldots, x_k]$ a polynomial system.

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Definition (Gröbner bases)

A $\succ$-*Gröbner basis* is a finite generating set G for $\langle F \rangle$ such that $\langle \mathsf{LM}_\succ(G) \rangle = \mathsf{LM}_\succ(\langle F \rangle)$.

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$g_1, \ldots, g_m$ is a $\succ$-Gröbner basis for $\langle g_1, \ldots, g_m \rangle$ if and only if all S-pairs reduce to zero upon division by $g_1, \ldots, g_m$.

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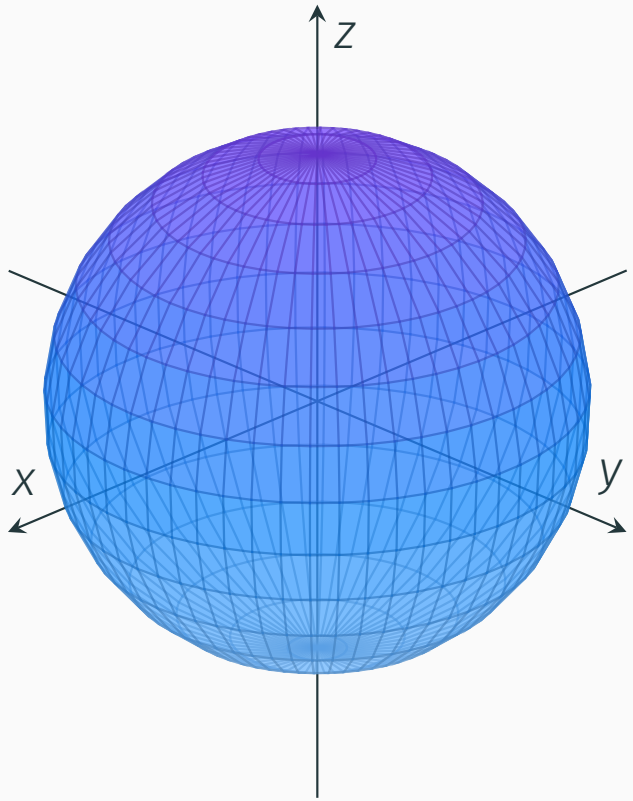
For zero-dimensional systems:

$$\max_{f \in F} \{\deg f\}^{O(\#\text{ of variables})}$$ [Lazard, 1983]

An example

$$f_1 = x^2 + y^2 + z^2 - 1$$



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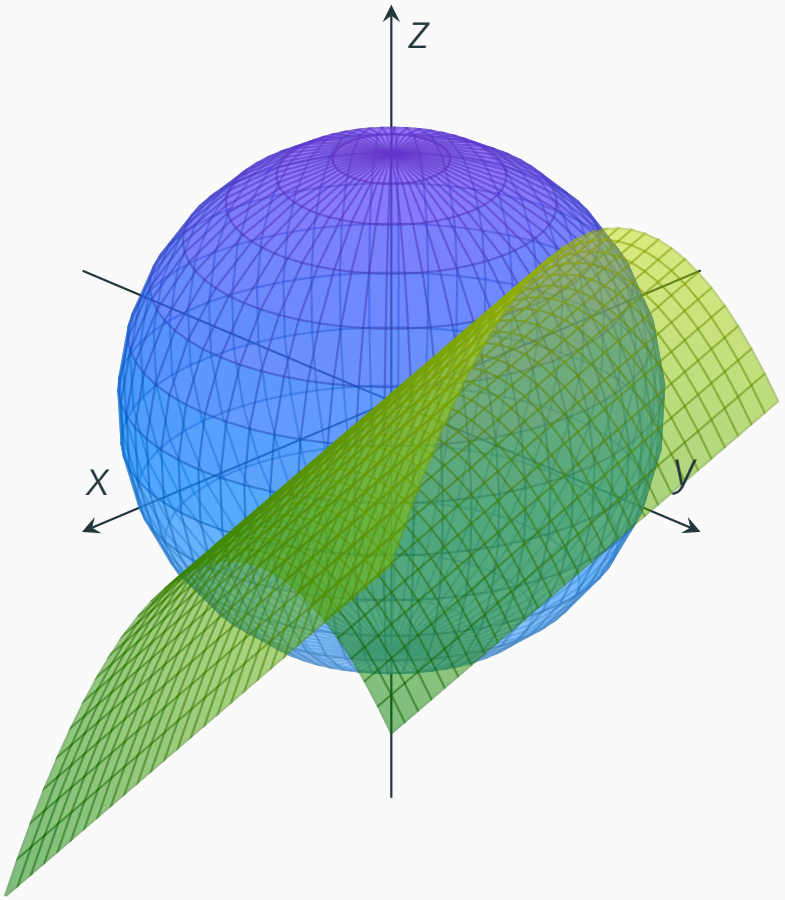

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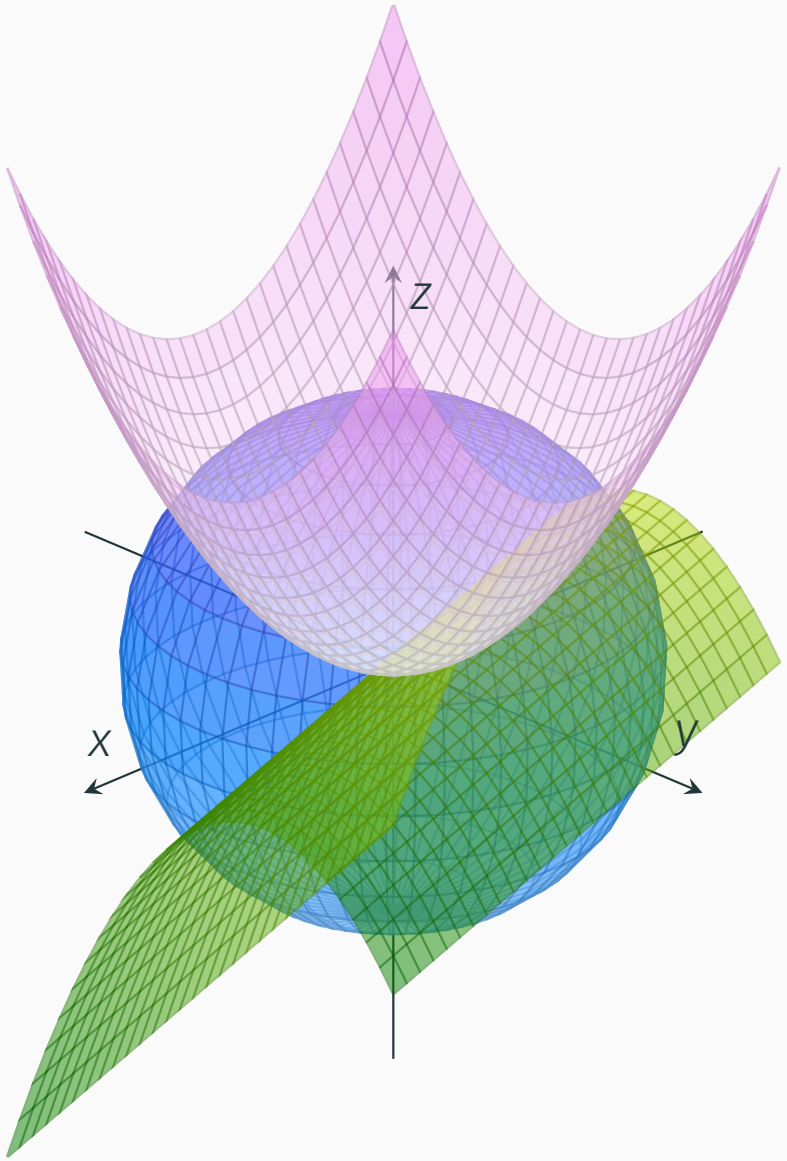

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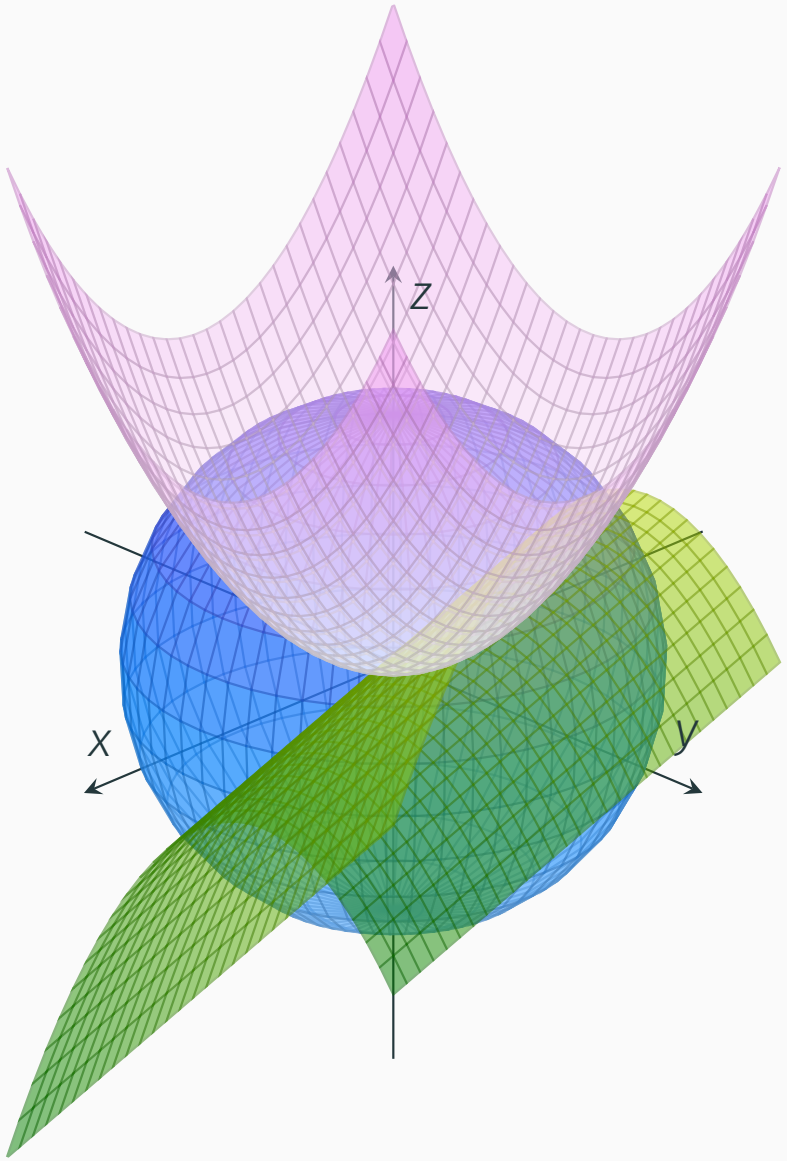

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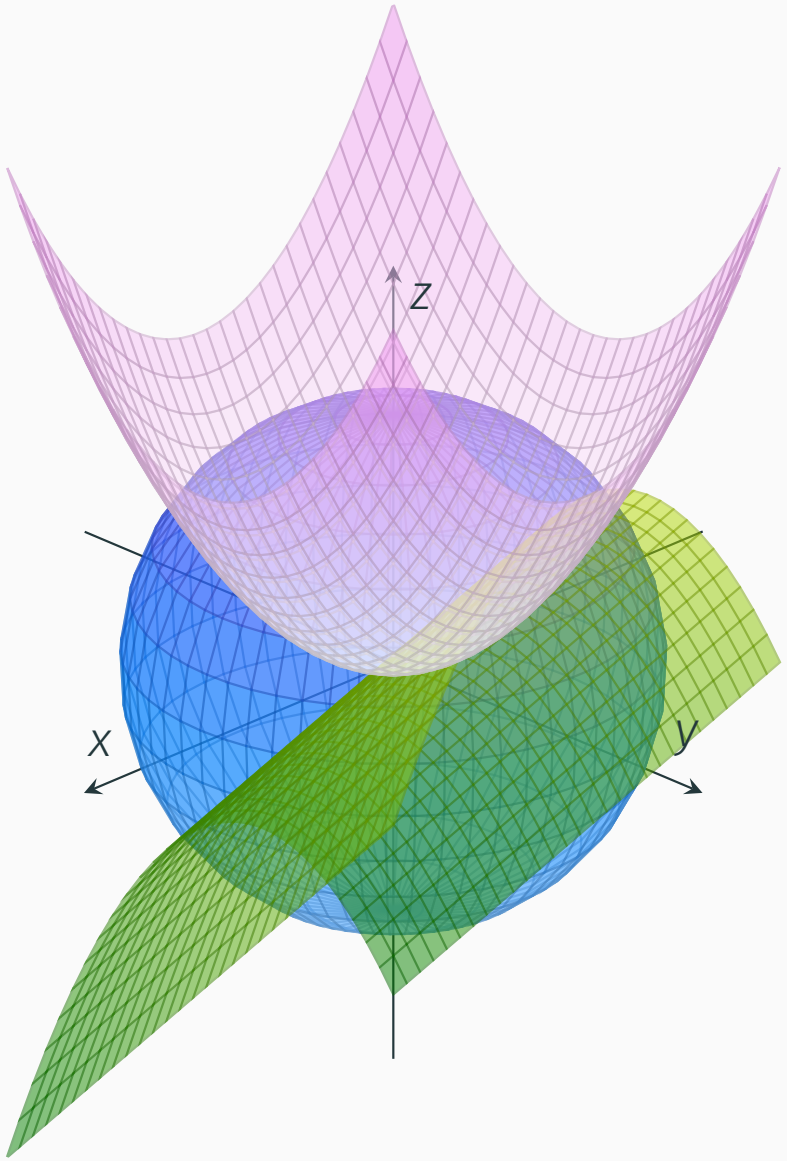

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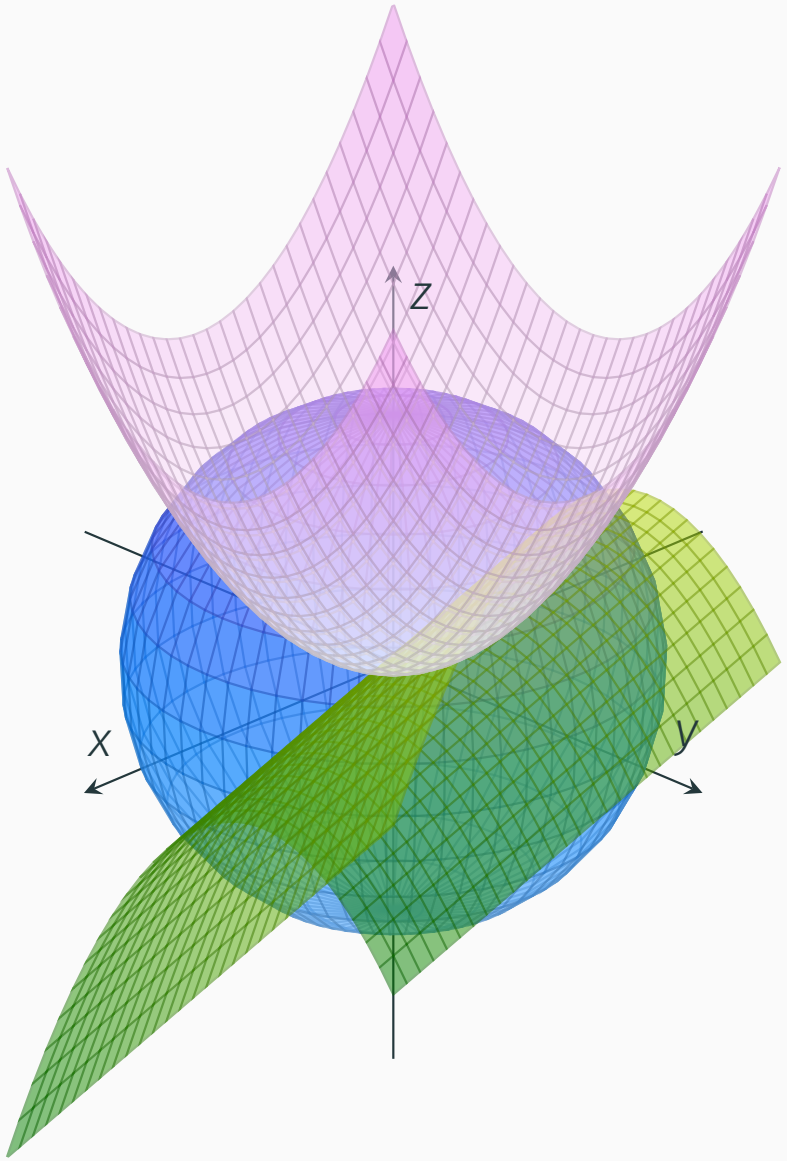

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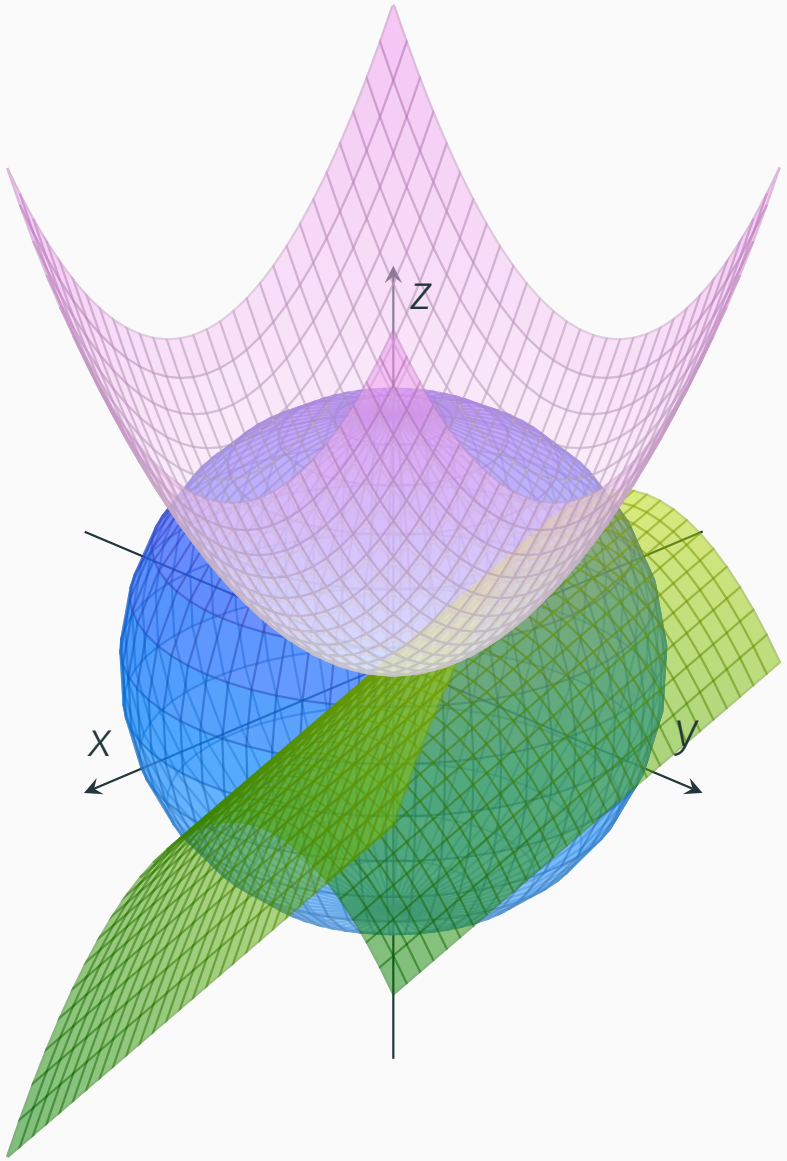

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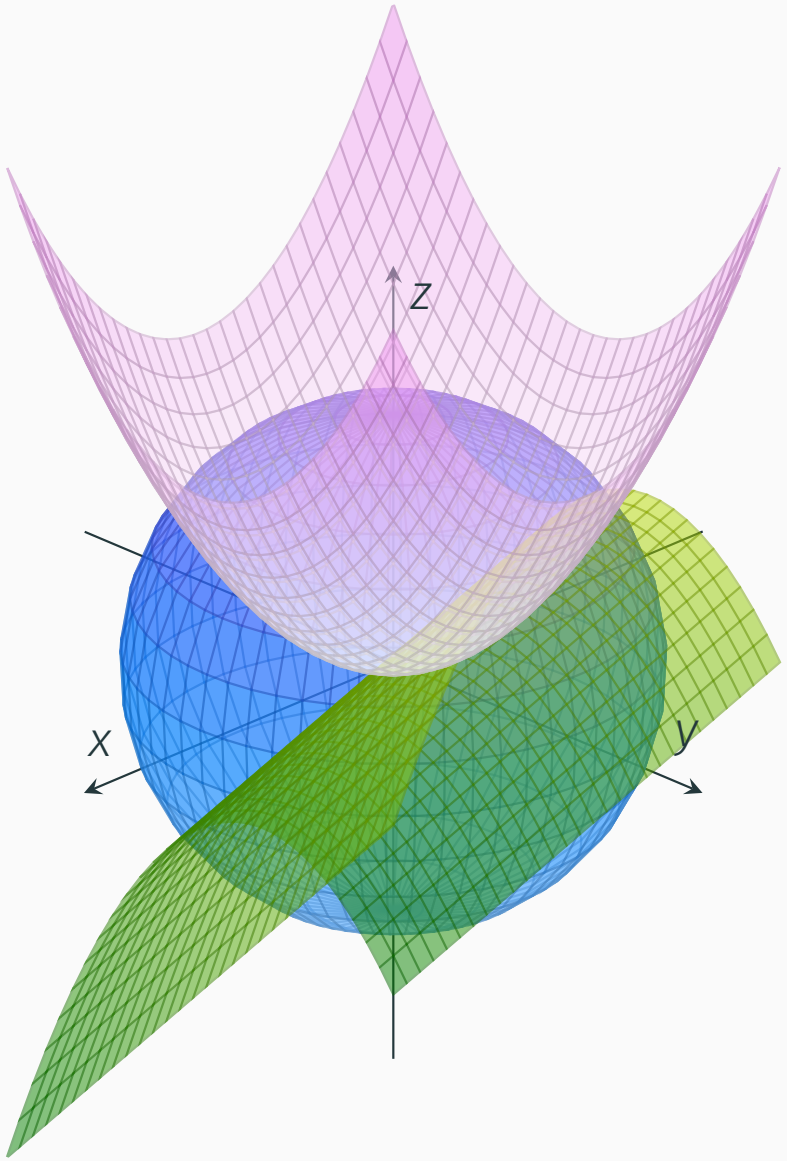

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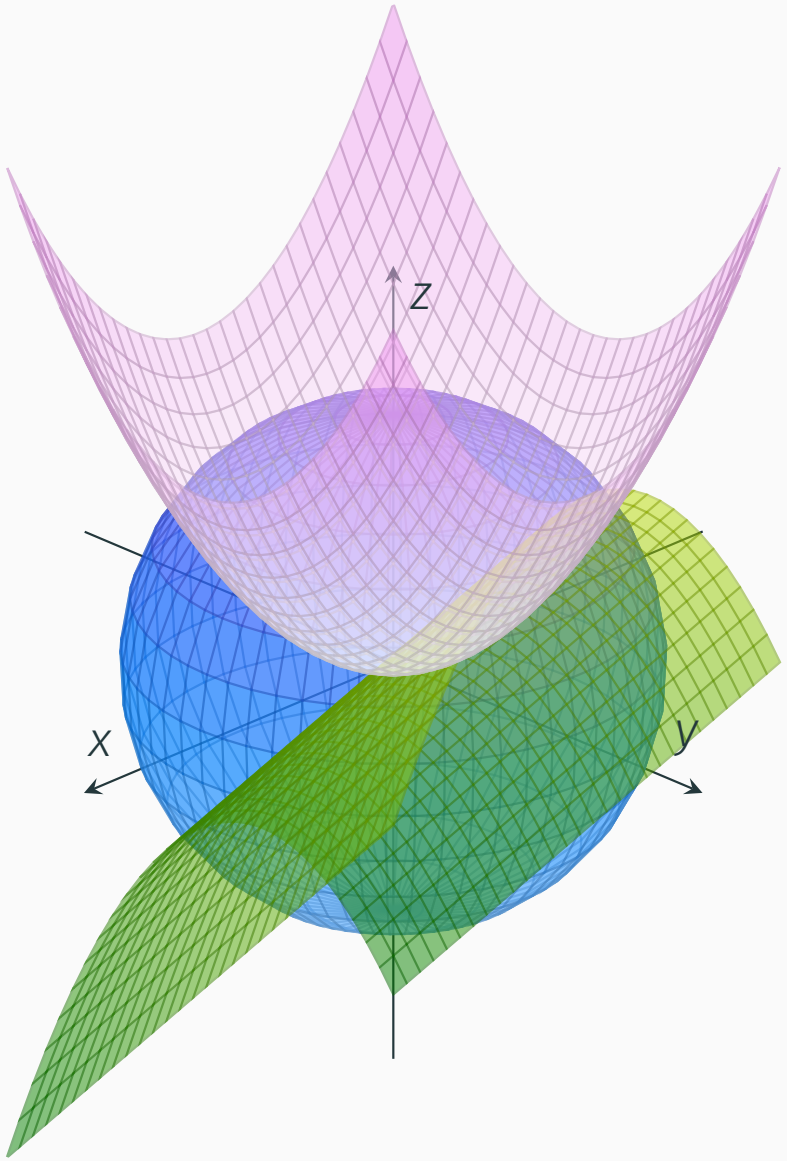

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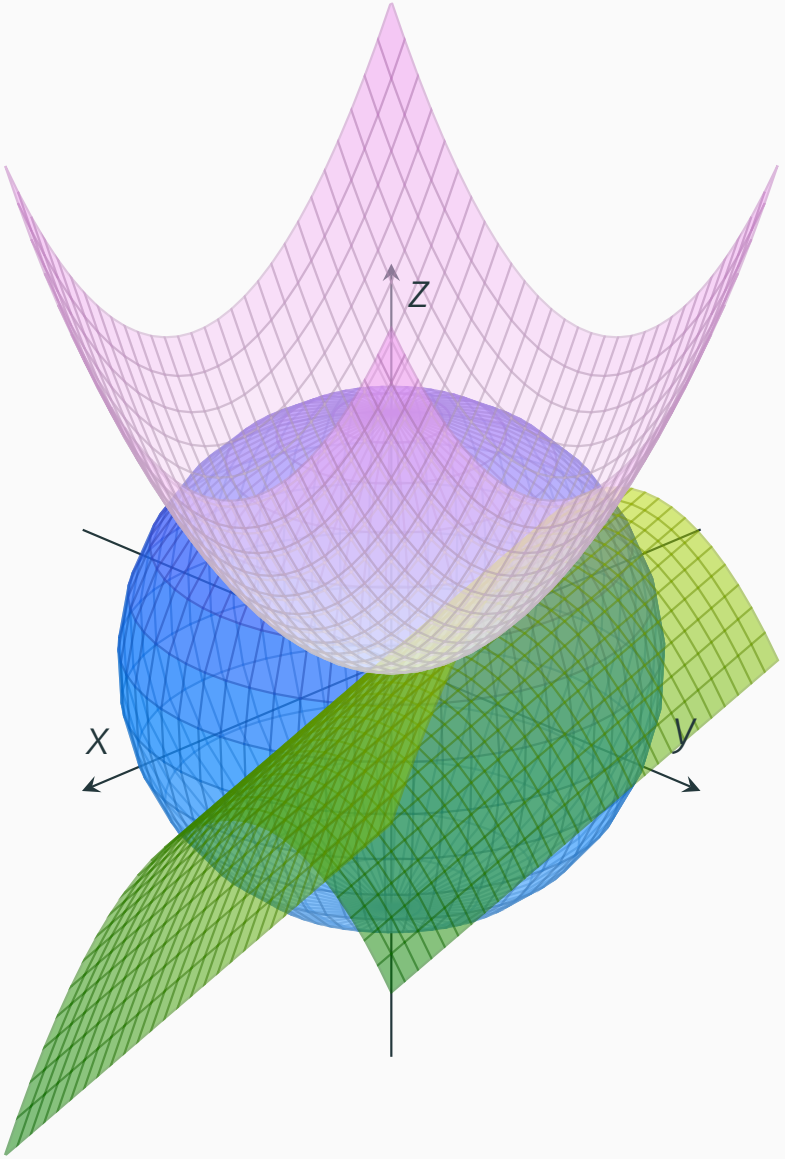

Macaulay matrices - linearization

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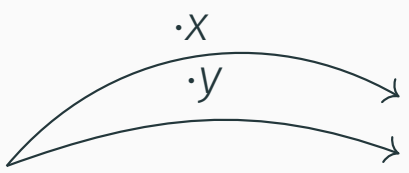

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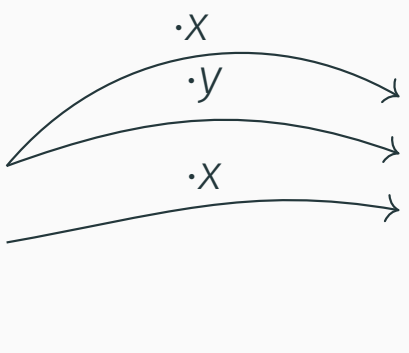

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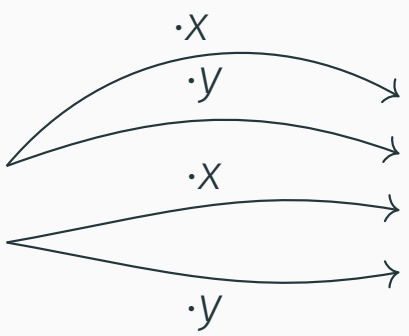

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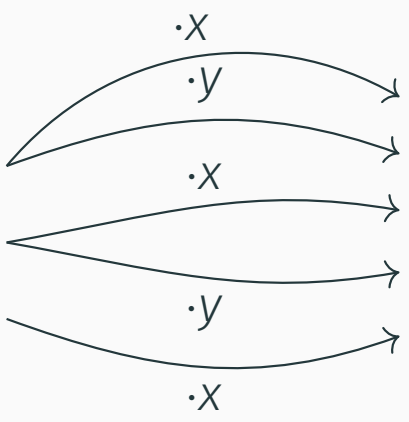

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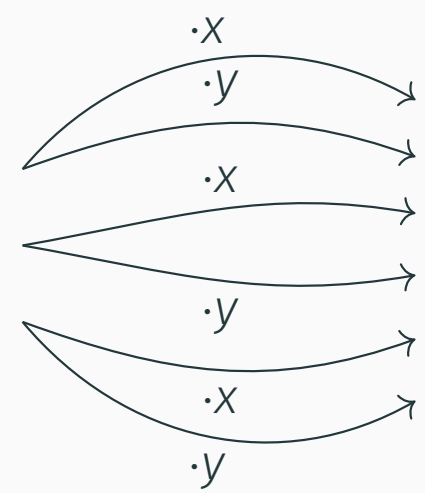

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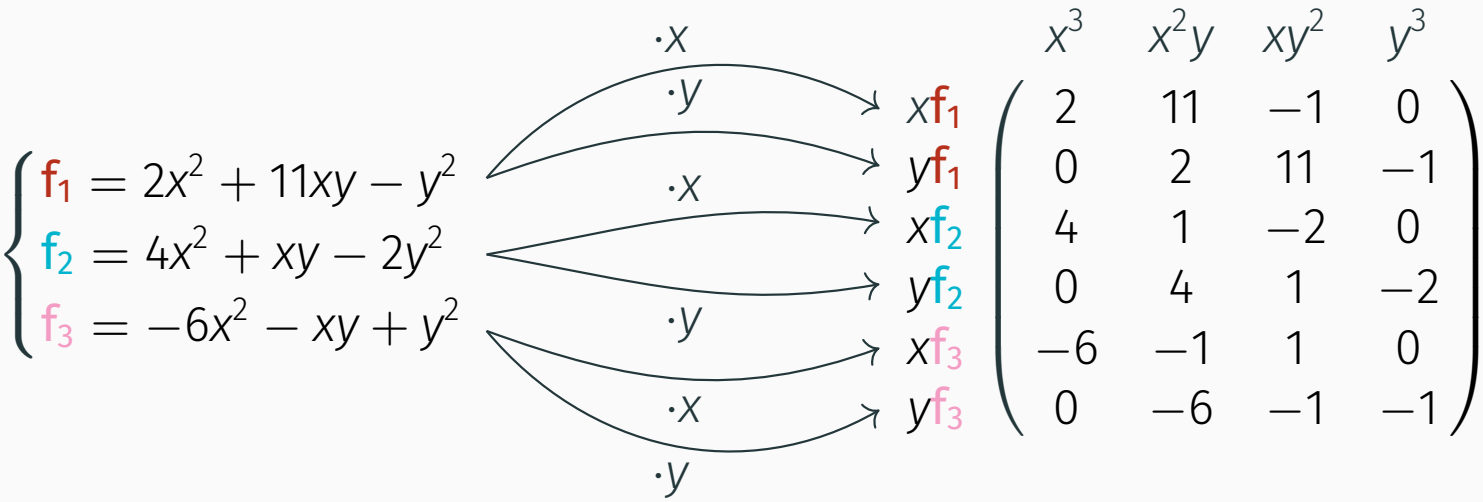

Theorem (Macaulay bound, [Lazard, 1983])

The maximum degree of a polynomial in the ***grevlex*** *Gröbner basis of a generic polynomial system $f_1, \ldots, f_m$ is*

$$\left(\sum_{i=1}^{m} \deg(f_i) - 1 \right) + 1.$$

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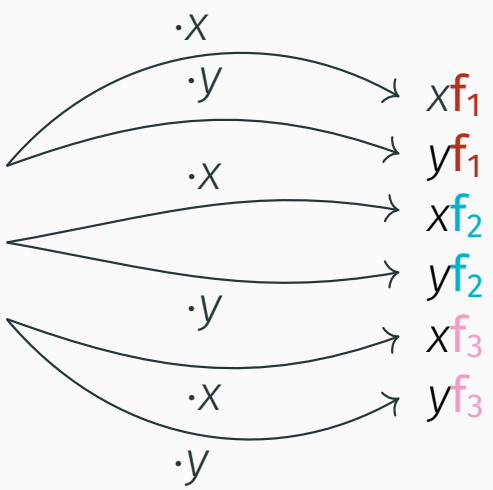

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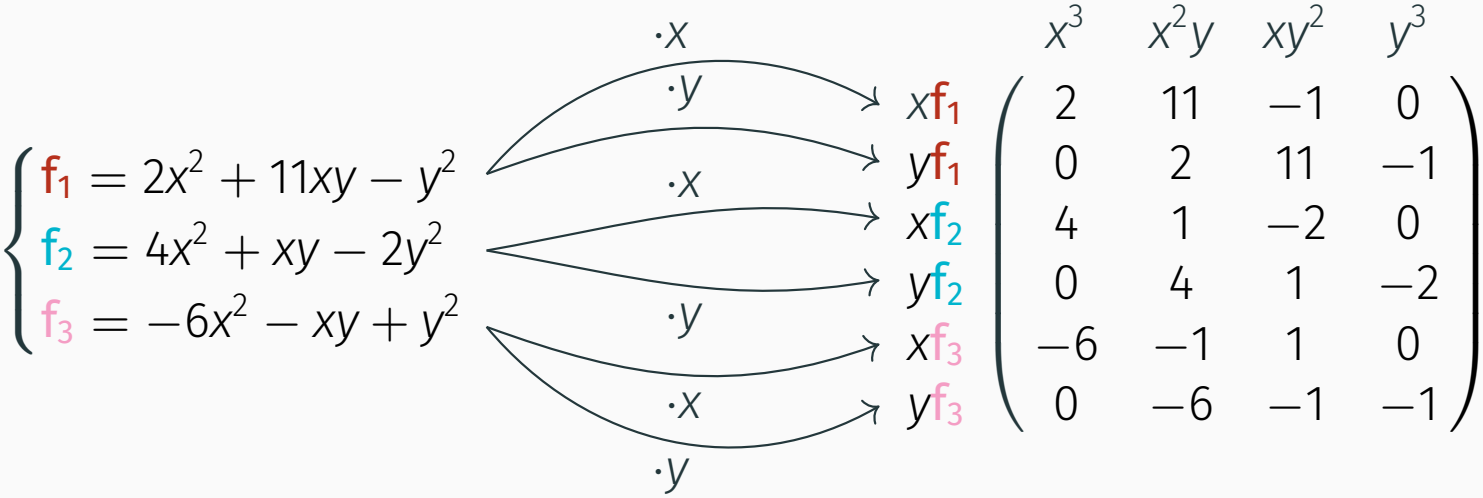

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The rows of the echelonization of the Macaulay matrix of F in degree d form the elements of degree d of a $\succ$-Gröbner basis for F.

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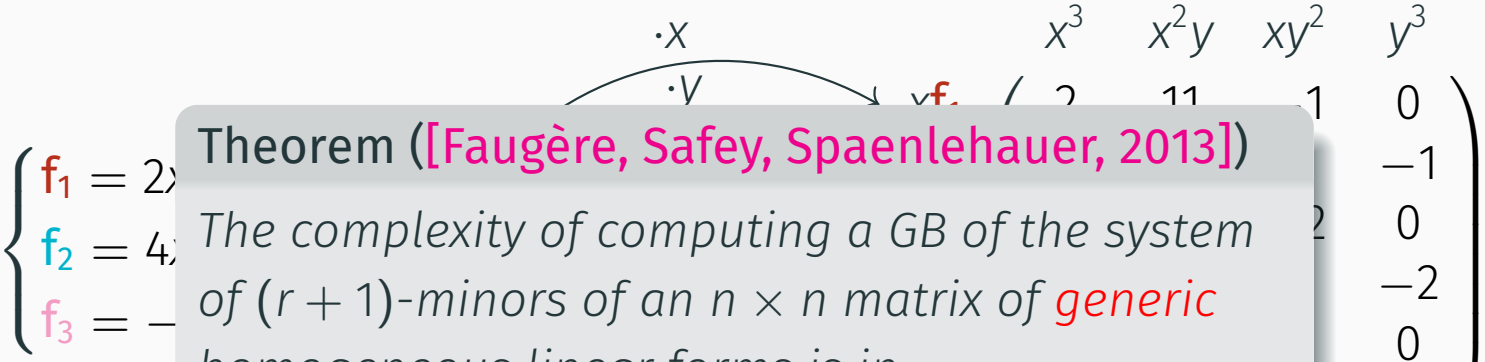


Theorem ([Faugère, Safey, Spaenlehauer, 2013])

The complexity of computing a GB of the system of $(r+1)$-minors of an $n \times n$ matrix of generic homogeneous linear forms is in

$$O\left(\binom{n}{r+1}^2 \cdot \binom{k+r(n-r)+1}{k}^{\omega}\right).$$

Theorem (Ma

*The maximum degree of a polynomial in the **grevlex** Gröbner basis of a generic polynomial system $f_1, \ldots, f_m$ is*

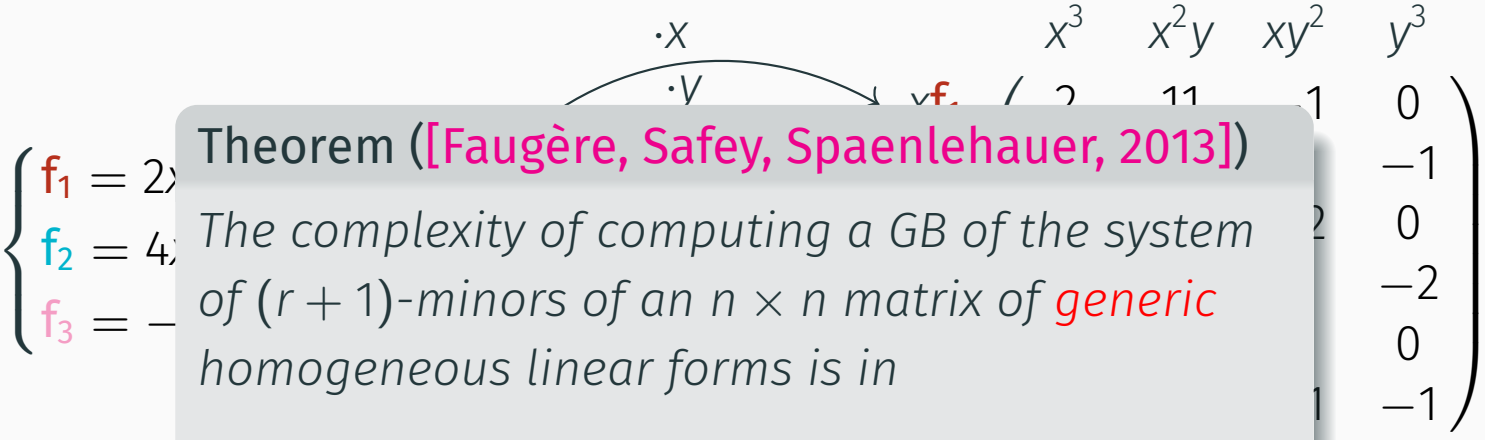
algebraic property: regularity

$$\left(\sum_{i=1}^{m} \deg(f_i) - 1\right) + 1.$$

The rows of the echelonization of the Macaulay matrix of F in degree d form the elements of degree d of a $\succ$-Gröbner basis for F.

Macaulay matrices - linearization

Assume F homogeneous.



Theorem ([Faugère, Safey, Spaenlehauer, 2013])

The complexity of computing a GB of the system of $(r+1)$-minors of an $n \times n$ matrix of generic homogeneous linear forms is in

$$O\left(\binom{n}{r+1}^2 \cdot \binom{k+r(n-r)+1}{k}^{\omega}\right).$$

Theorem (Ma

*The maximum degree of a polynomial in the **grevlex** Gröbner basis of a generic polynomial system $f_1, \ldots, f_m$ is*

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This upper bound is not very sharp since the Macaulay matrices are not full rank!

The rows of the echelonization of the Macaulay matrix of F in degree d form the elements of degree d of a $\succ$-Gröbner basis for F.

The F_5 algorithm ([Faugère, 2002])

Let $\mathbb{k} = \mathbb{F}_7$, $\succ =$ grevlex .

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$$f_1 = 5x^2 + 5xy + 3y^2 + 5xz + 5yz + 6z^2$$
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$$\widetilde{\mathcal{M}_2}$$

$$\begin{array}{c} \\ (1,1) \\ (2,1) \\ (3,1) \end{array} \begin{array}{c} \begin{array}{cccccc} x^2 & xy & y^2 & xz & yz & z^2 \end{array} \\ \begin{pmatrix} 1 & 1 & 2 & 1 & 1 & 4 \\ 0 & 1 & 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 2 & 0 & 4 \end{pmatrix} \end{array}$$

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$\widetilde{\mathcal{M}_2}$

	x^2	xy	y^2	xz	yz	z^2
$(1,1)$	1	1	2	1	1	4
$(2,1)$	0	1	0	0	2	4
$(3,1)$	0	0	1	2	0	4

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
$(1,x^2)$	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
$(1,xy)$	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
$(1,y^2)$	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
$(1,xz)$	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
$(1,yz)$	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
$(1,z^2)$	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
$(2,x^2)$	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
$(2,xy)$	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
$(2,y^2)$	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
$(2,yz)$	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
$(2,z^2)$	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
$(3,x^2)$	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
$(3,xy)$	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
$(3,y^2)$	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
$(3,xz)$	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
$(3,yz)$	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
$(3,z^2)$	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

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$\widetilde{\mathcal{M}_2}$

	x^2	xy	y^2	xz	yz	z^2
$(1,1)$	1	1	2	1	1	4
$(2,1)$	0	1	0	0	2	4
$(3,1)$	0	0	1	2	0	4

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
$(1,x^2)$	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
$(1,xy)$	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
$(1,y^2)$	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
$(1,xz)$	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
$(1,yz)$	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
$(1,z^2)$	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
$(2,x^2)$	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
$(2,xy)$	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
$(2,y^2)$	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
$(2,yz)$	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
$(2,z^2)$	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
$(3,x^2)$	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
$(3,xy)$	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
$(3,y^2)$	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
$(3,xz)$	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
$(3,yz)$	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
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$\widetilde{\mathcal{M}_2}$

	x^2	xy	y^2	xz	yz	z^2
$(1,1)$	1	1	2	1	1	4
$(2,1)$	0	1	0	0	2	4
$(3,1)$	0	0	1	2	0	4

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
$(1,x^2)$	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
$(1,xy)$	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
$(1,y^2)$	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
$(1,xz)$	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
$(1,yz)$	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
$(1,z^2)$	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
$(2,x^2)$	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
$(2,xy)$	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
$(2,y^2)$	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
$(2,yz)$	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
$(2,z^2)$	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
$(3,x^2)$	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
$(3,xy)$	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
$(3,y^2)$	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
$(3,xz)$	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
$(3,yz)$	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
$(3,z^2)$	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

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$(3,1)$	0	0	1	2	0	4

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
$(1,x^2)$	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
$(1,xy)$	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
$(1,y^2)$	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
$(1,xz)$	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
$(1,yz)$	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
$(1,z^2)$	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
$(2,x^2)$	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
$(2,xy)$	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
$(2,y^2)$	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
$(2,yz)$	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
$(2,z^2)$	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
$(3,x^2)$	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
$(3,xy)$	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
$(3,y^2)$	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
$(3,xz)$	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
$(3,yz)$	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
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$(3,1)$	0	0	1	2	0	4

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
$(1,x^2)$	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
$(1,xy)$	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
$(1,y^2)$	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
$(1,xz)$	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
$(1,yz)$	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
$(1,z^2)$	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
$(2,x^2)$	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
$(2,xy)$	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
$(2,y^2)$	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
$(2,yz)$	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
$(2,z^2)$	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
$(3,x^2)$	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
$(3,xy)$	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
$(3,y^2)$	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
$(3,xz)$	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
$(3,yz)$	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
$(3,z^2)$	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

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$\widetilde{\mathcal{M}}_2$

	x^2	xy	y^2	xz	yz	z^2
$(1,1)$	1	1	2	1	1	4
$(2,1)$	0	1	0	0	2	4
$(3,1)$	0	0	1	2	0	4

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
$(1,x^2)$	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
$(1,xy)$	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
$(1,y^2)$	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
$(1,xz)$	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
$(1,yz)$	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
$(1,z^2)$	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
$(2,x^2)$	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
$(2,xy)$	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
$(2,y^2)$	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
$(2,yz)$	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
$(2,z^2)$	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
$(3,x^2)$	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
$(3,xy)$	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
$(3,y^2)$	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
$(3,xz)$	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
$(3,yz)$	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
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$\widetilde{\mathcal{M}_2}$

	x^2	xy	y^2	xz	yz	z^2
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$(2,1)$	0	1	0	0	2	4
$(3,1)$	0	0	1	2	0	4

Lazard: $\mathcal{M}_4$ is 18×15.

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
$(1,x^2)$	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
$(1,xy)$	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
$(1,y^2)$	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
$(1,xz)$	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
$(1,yz)$	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
$(1,z^2)$	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
$(2,x^2)$	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
$(2,xy)$	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
$(2,y^2)$	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
$(2,yz)$	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
$(2,z^2)$	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
$(3,x^2)$	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
$(3,xy)$	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
$(3,y^2)$	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
$(3,xz)$	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
$(3,yz)$	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
$(3,z^2)$	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

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Lazard: $\mathcal{M}_4$ is 18×15.

F5: $\mathcal{M}_4$ is 15×15 and full rank!

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
$(1,x^2)$	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
$(1,xy)$	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
$(1,y^2)$	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
$(1,xz)$	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
$(1,yz)$	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
$(1,z^2)$	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
$(2,x^2)$	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
$(2,xy)$	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
$(2,y^2)$	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
$(2,yz)$	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
$(2,z^2)$	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
$(3,x^2)$	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
$(3,xy)$	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
$(3,y^2)$	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
$(3,xz)$	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
$(3,yz)$	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
$(3,z^2)$	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

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$(f_1, \ldots, f_m)$ generic

$\widetilde{\mathcal{M}_2}$

	x^2	xy	y^2	xz	yz	z^2
$(1,1)$	1	1	2	1	1	4
$(2,1)$	0	1	0	0	2	4
$(3,1)$	0	0	1	2	0	4

Lazard: $\mathcal{M}_4$ is 18×15.

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	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
$(1,x^2)$	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
$(1,xy)$	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
$(1,y^2)$	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
$(1,xz)$	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
$(1,yz)$	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
$(1,z^2)$	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
$(2,x^2)$	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
$(2,xy)$	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
$(2,y^2)$	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
$(2,yz)$	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
$(2,z^2)$	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
$(3,x^2)$	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
$(3,xy)$	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
$(3,y^2)$	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
$(3,xz)$	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
$(3,yz)$	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
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$\Downarrow$

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$(1,1)$	1	1	2	1	1	4
$(2,1)$	0	1	0	0	2	4
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	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
$(1,x^2)$	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
$(1,xy)$	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
$(1,y^2)$	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
$(1,xz)$	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
$(1,yz)$	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
$(1,z^2)$	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
$(2,x^2)$	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
$(2,xy)$	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
$(2,y^2)$	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
$(2,yz)$	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
$(2,z^2)$	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
$(3,x^2)$	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
$(3,xy)$	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
$(3,y^2)$	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
$(3,xz)$	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
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$(f_1, \ldots, f_m)$ generic

$\Downarrow$

No reductions to zero.

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	x^2	xy	y^2	xz	yz	z^2
$(1,1)$	1	1	2	1	1	4
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	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
$(1,x^2)$	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
$(1,xy)$	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
$(1,y^2)$	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
$(1,xz)$	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
$(1,yz)$	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
$(1,z^2)$	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
$(2,x^2)$	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
$(2,xy)$	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
$(2,y^2)$	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
$(2,yz)$	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
$(2,z^2)$	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
$(3,x^2)$	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
$(3,xy)$	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
$(3,y^2)$	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
$(3,xz)$	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
$(3,yz)$	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
$(3,z^2)$	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

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$(f_1, \ldots, f_m)$ generic

$\Downarrow$

- **No** reductions to zero.
- Precise complexity analysis [1]

$\widetilde{\mathcal{M}_2}$

	x^2	xy	y^2	xz	yz	z^2
$(1,1)$	1	1	2	1	1	4
$(2,1)$	0	1	0	0	2	4
$(3,1)$	0	0	1	2	0	4

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$(1,x^2)$	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
$(1,xy)$	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
$(1,y^2)$	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
$(1,xz)$	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
$(1,yz)$	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
$(1,z^2)$	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
$(2,x^2)$	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
$(2,xy)$	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
$(2,y^2)$	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
$(2,yz)$	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
$(2,z^2)$	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
$(3,x^2)$	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
$(3,xy)$	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
$(3,y^2)$	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
$(3,xz)$	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
$(3,yz)$	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
$(3,z^2)$	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

[1] [Bardet, Faugère, Salvy, 2015]

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$$\widetilde{\mathcal{M}_2}$$

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$\Downarrow$

zero.

y analysis [1]

Determinantal systems are not generic!

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$(1,x^2)$	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
$(1,xy)$	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
$(1,y^2)$	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
$(1,xz)$	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
$(1,yz)$	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
$(1,z^2)$	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
$(2,x^2)$	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
$(2,xy)$	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
$(2,y^2)$	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
$(2,yz)$	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
$(2,z^2)$	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
$(3,x^2)$	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
$(3,xy)$	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
$(3,y^2)$	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
$(3,xz)$	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
$(3,yz)$	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
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$$\widetilde{\mathcal{M}_2}$$

$(f_1, \ldots, f_m)$ generic

$\Downarrow$

zero.

y analysis [1]

Determinantal systems are not generic!

How do we remove reductions to zero?

	x^2	xy	y^2	xz	yz	z^2
$(1,1)$	1	1	2	1	1	4
$(2,1)$	0	1	0			
$(3,1)$	0	0	1			

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
$(1,x^2)$	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
$(1,xy)$	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
$(1,y^2)$	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
									5	5	0	6	0	0	
									0	5	5	0	6	0	
$(1,z^2)$	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
$(2,x^2)$	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
$(2,xy)$	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
$(2,y^2)$	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
$(2,yz)$	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
$(2,z^2)$	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
$(3,x^2)$	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
$(3,xy)$	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
$(3,y^2)$	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
$(3,xz)$	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
$(3,yz)$	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
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Contributions ([G., Neiger, Safey El Din, 2023])

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M is an $n \times n$ matrix of generic linear forms over $\mathbb{k}[x_1, \ldots, x_k]$, $r \leq n-1$. Let $F_r(M)$ be the system of $(r+1)$-minors of M. Suppose $F_r(M)$ is zero-dimensional.

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New F_5-type criteria

- Allows us to avoid all reductions to zero in degree $r+2$

$$\rightsquigarrow \binom{n}{r+2}^2 \left(\frac{2(r+2)(r+1)}{n-r-1} + 2r + 2 \right) \text{ reductions avoided.}$$

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Theorem ([G., Neiger, Safey El Din, 2023])

The complexity of computing a grevlex Gröbner basis for the system of $(n-1)$-minors of M is in

Homogeneous: $O(n^{4\omega-1})$

Affine: $O(n^{4\omega})$

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New F_5-type criteria

- Allows us to avoid all reductions to zero in degree $r+2$

$$\rightsquigarrow \binom{n}{r+2}^2 \left(\frac{2(r+2)(r+1)}{n-r-1} + 2r + 2 \right) \text{ reductions avoided.}$$

- When $r = n-2$, allows us to avoid **all** reductions to zero.

Theorem ([G., Neiger, Safey El Din, 2023])

The complexity of computing a grevlex Gröbner basis for the system of $(n-1)$-minors of M is in

Homogeneous: $O(n^{4\omega-1}) \rightsquigarrow O(n^{2\omega+3})$ *and* $|GB| \in \Omega(n^6)$ *conjecturally*

Affine: $O(n^{4\omega})$

Syzygies

Definition (Syzygy)

$(a_1, \ldots, a_m) \in \Bbbk[x_1, \ldots, x_k]^m$ with

$$a_1 f_1 + \cdots + a_m f_m = 0$$

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Theorem ([Hilbert, 1890])

Free resolution $0 \to \mathcal{E}_\ell \xrightarrow{d_\ell} \mathcal{E}_{\ell-1} \xrightarrow{d_{\ell-1}} \cdots \to \mathcal{E}_1 \xrightarrow{d_1} \mathcal{E}_0 \xrightarrow{\epsilon} \langle F \rangle \to 0 \implies$

$$\mathrm{Syz}_k(F) = \ker(d_k) = \mathrm{im}(d_{k+1}).$$

The Gulliksen-Negård complex

m_{ij} = determinant of submatrix of M given by deleting i-th row, j-th column.

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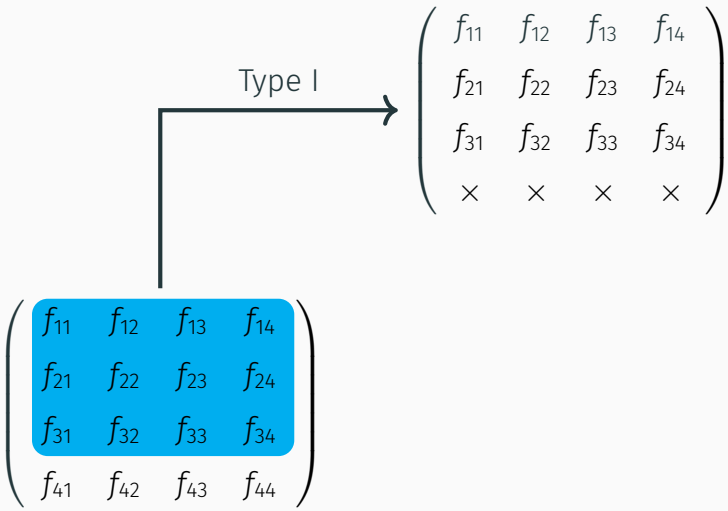

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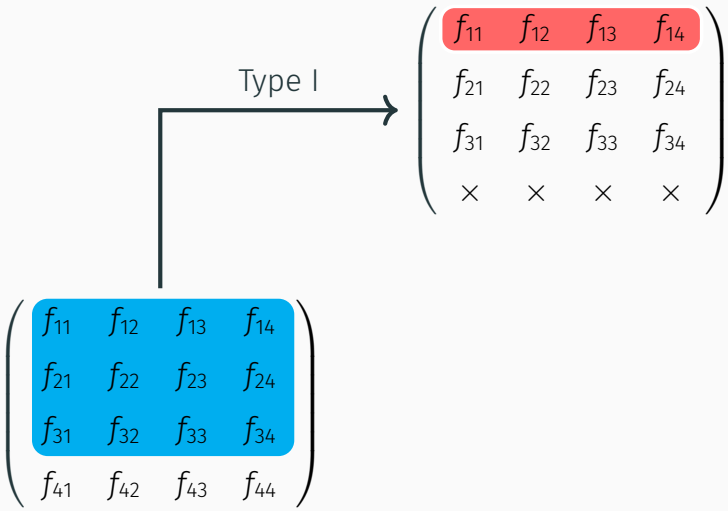

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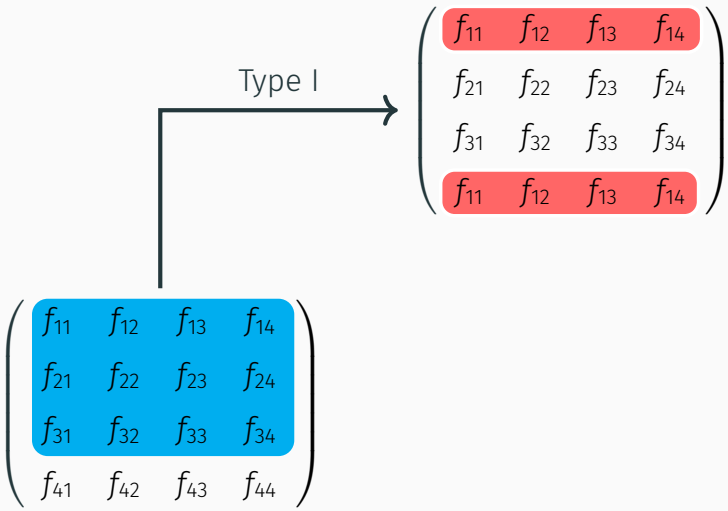

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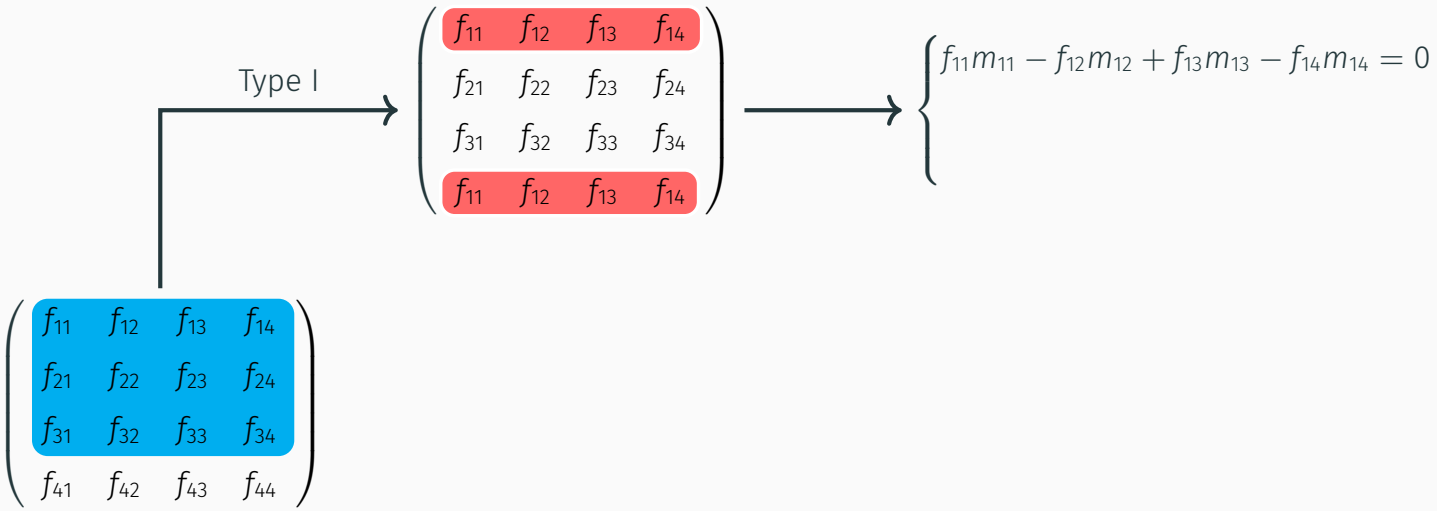

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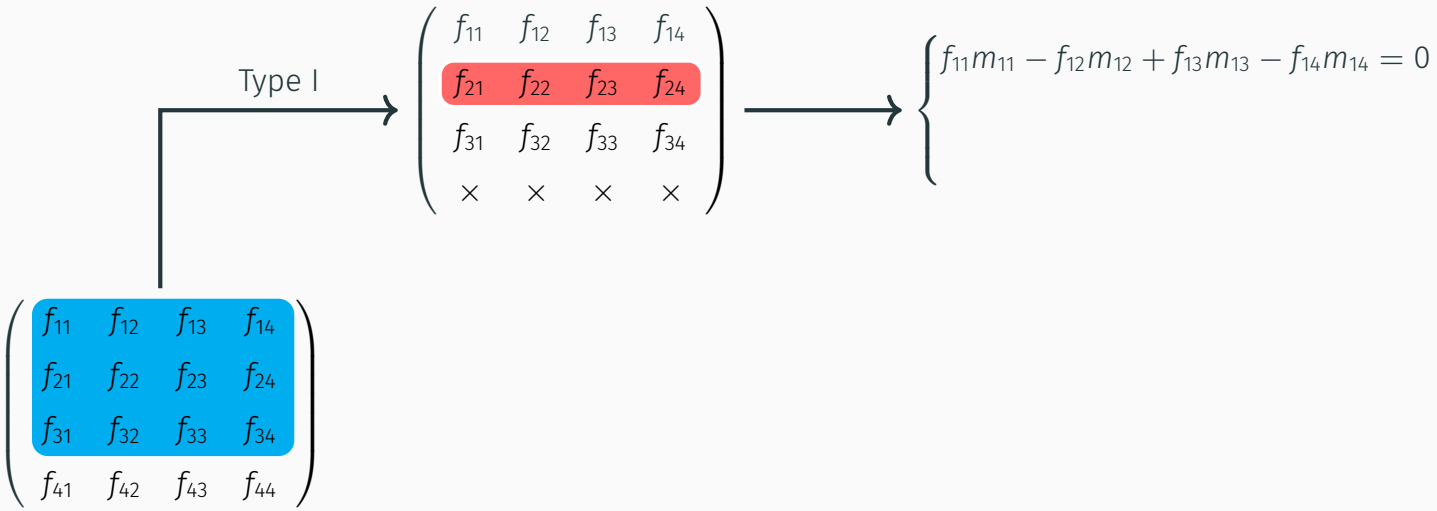

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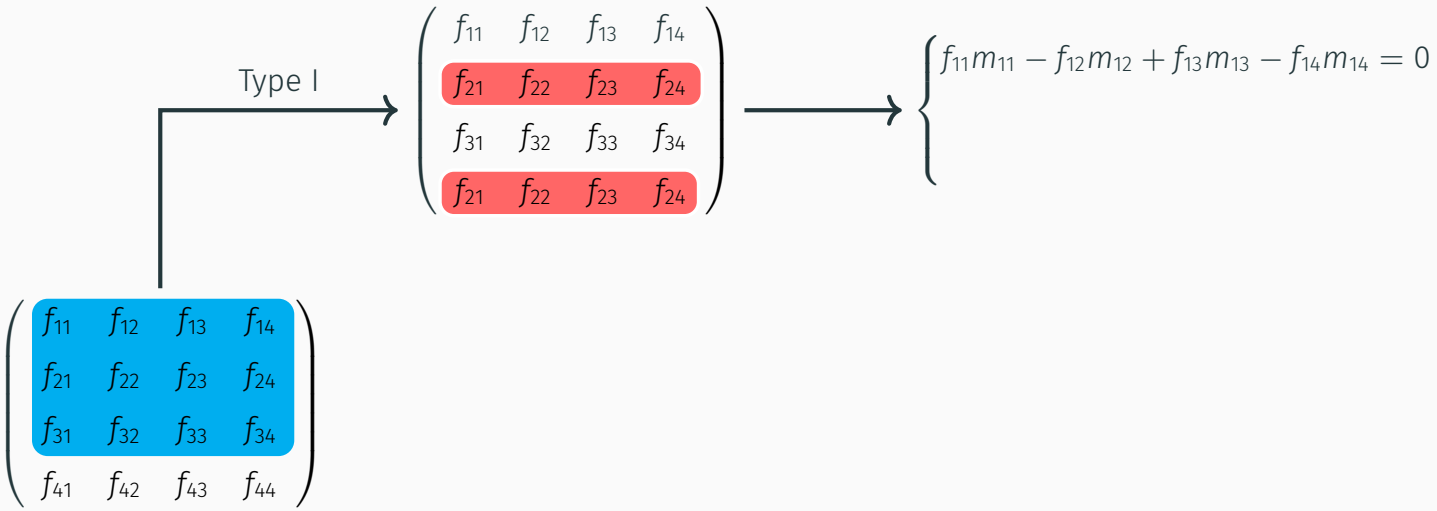

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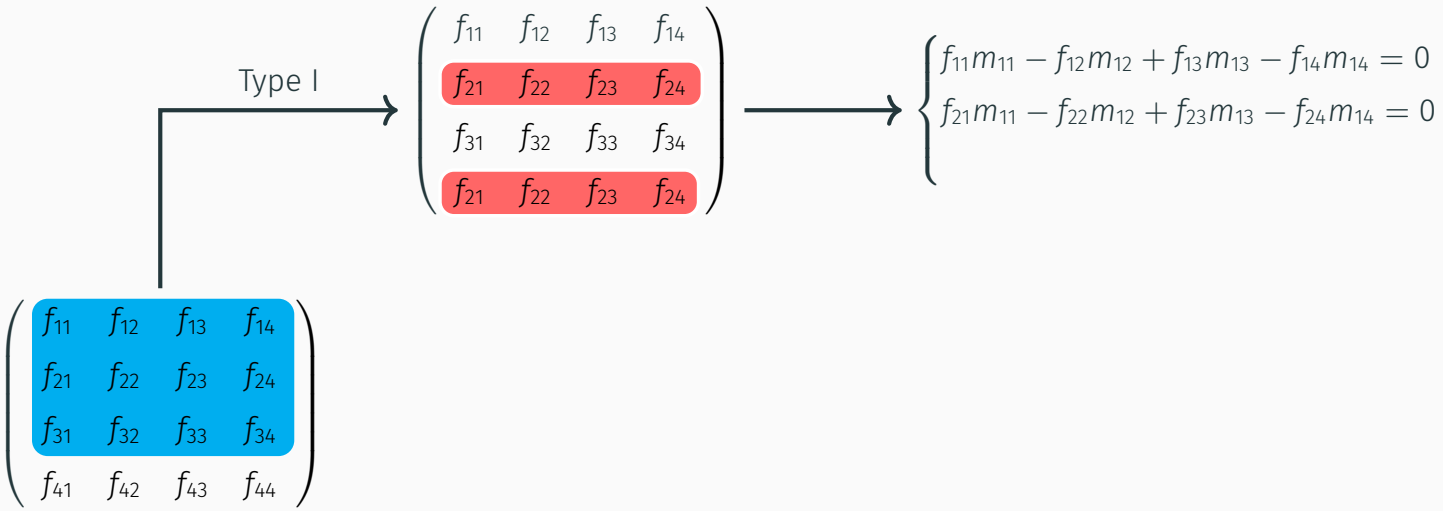

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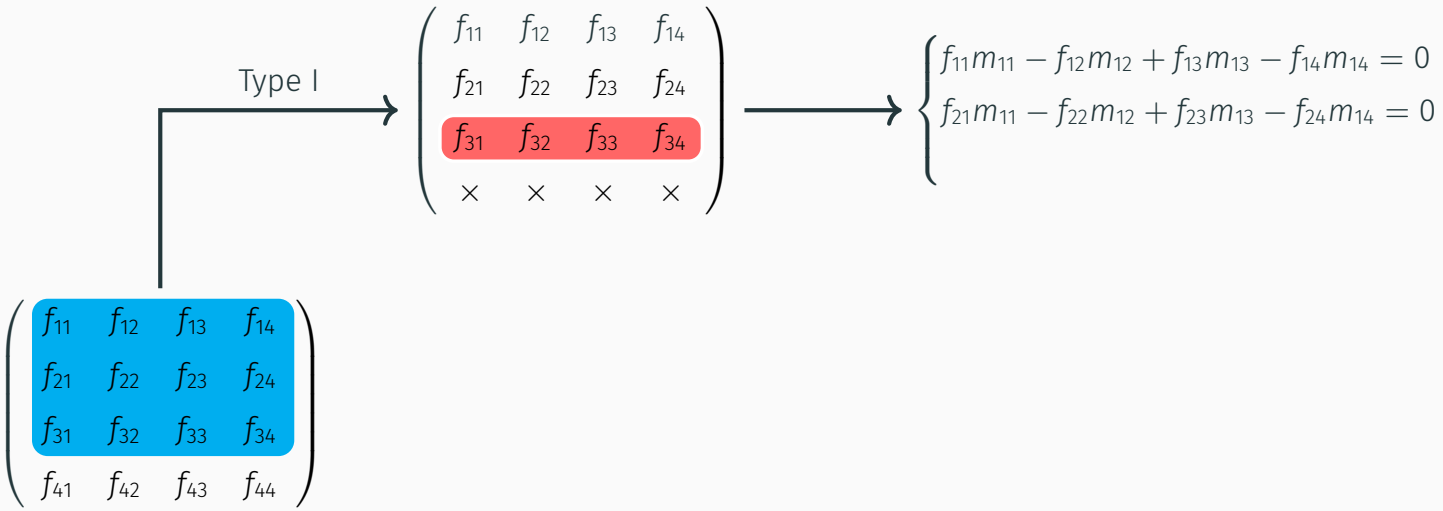

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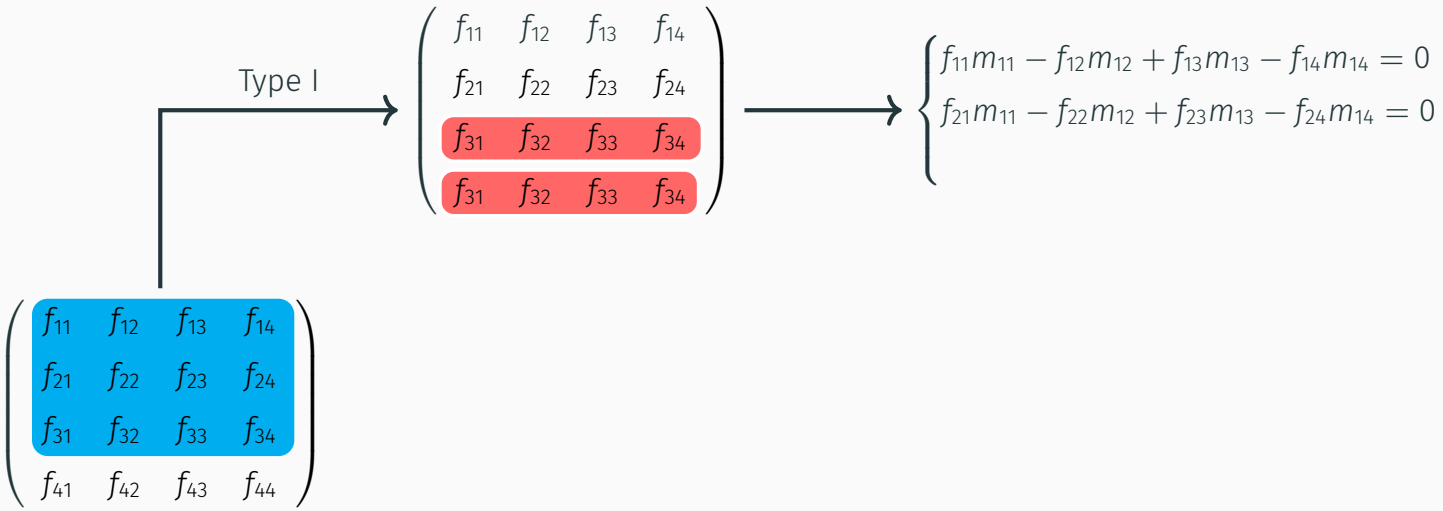

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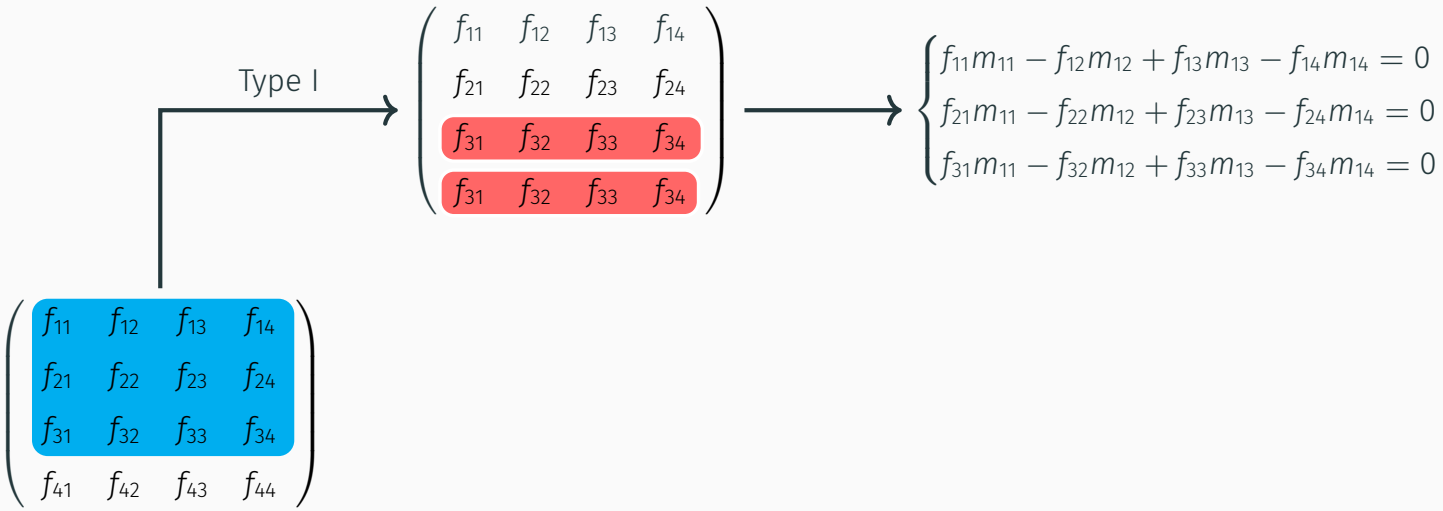

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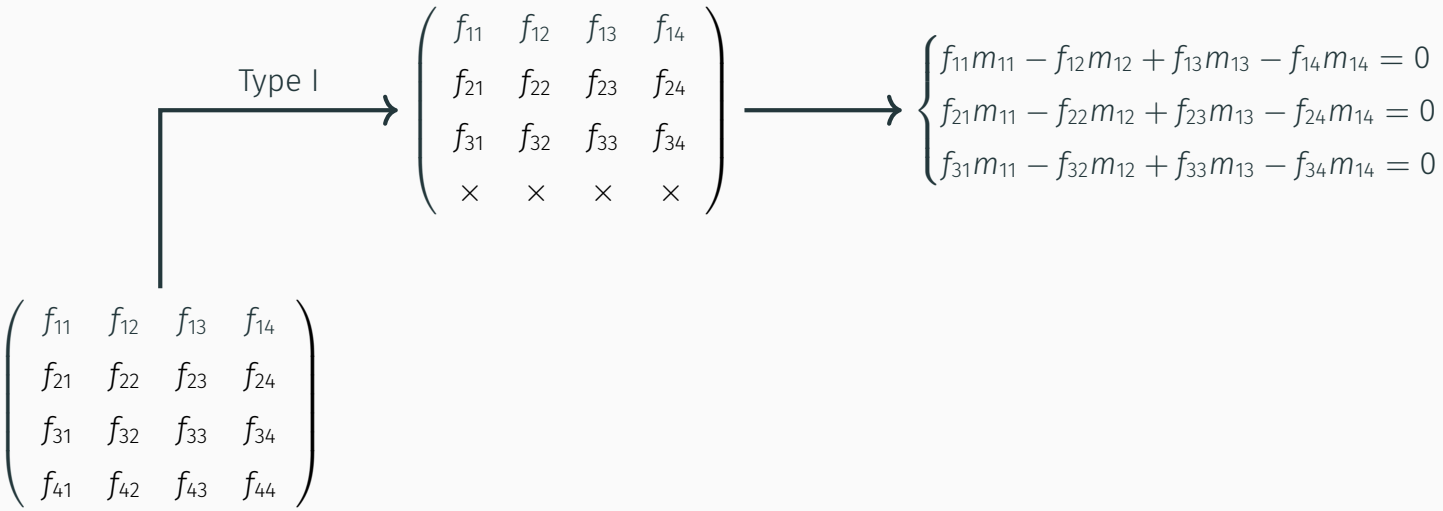

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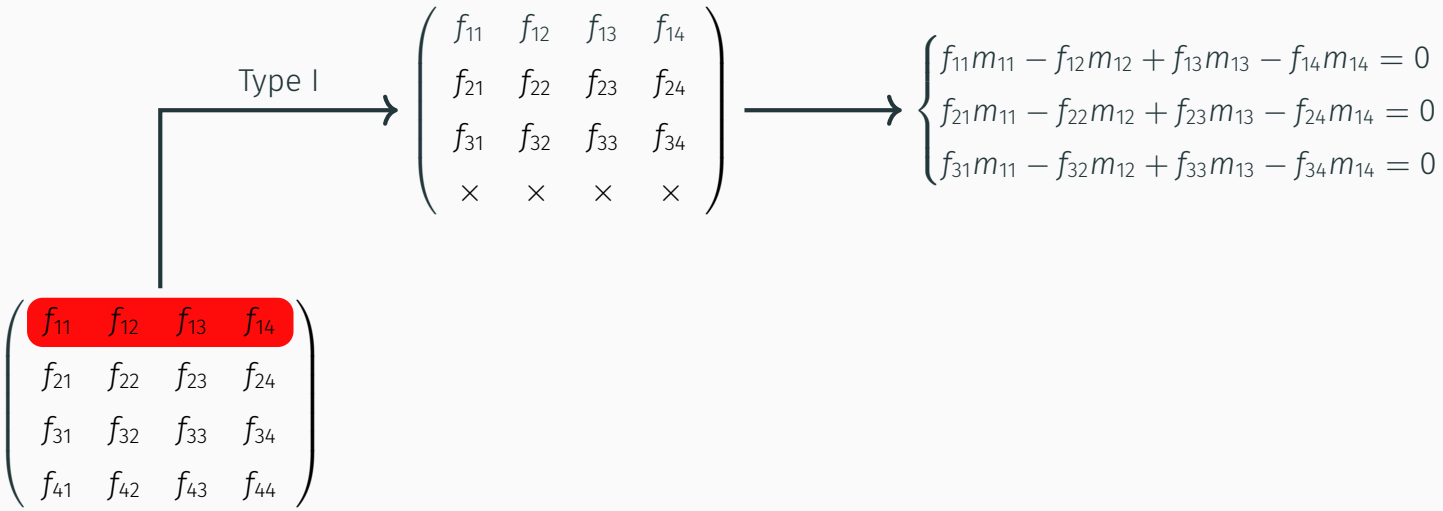

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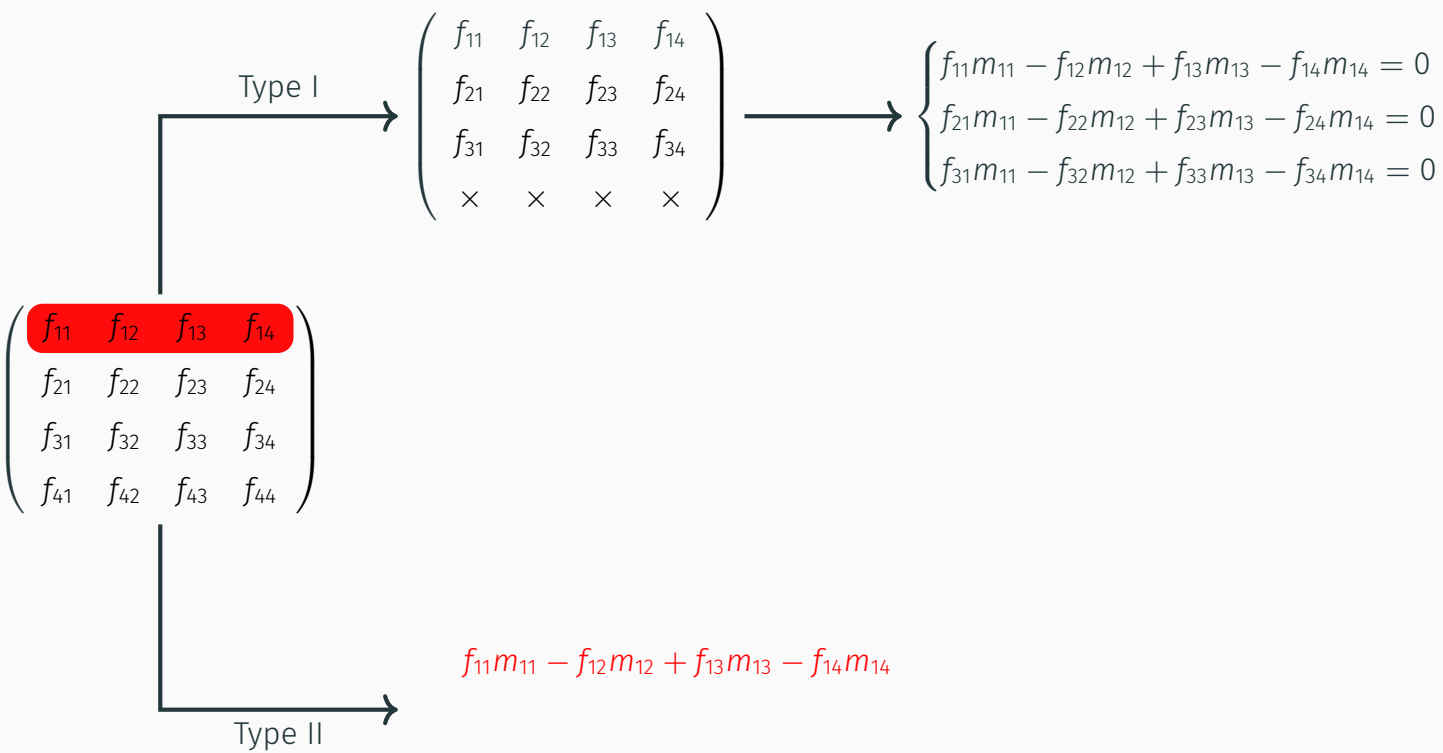

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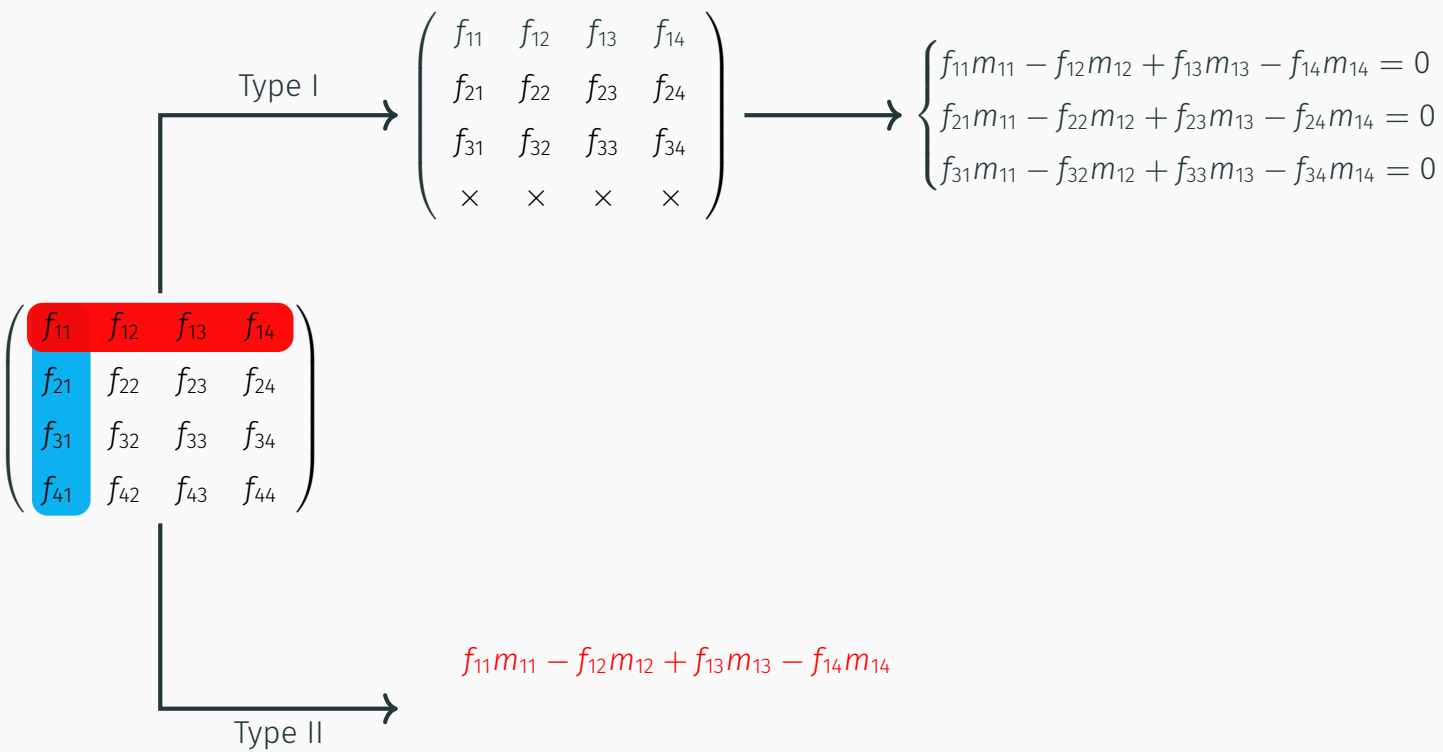

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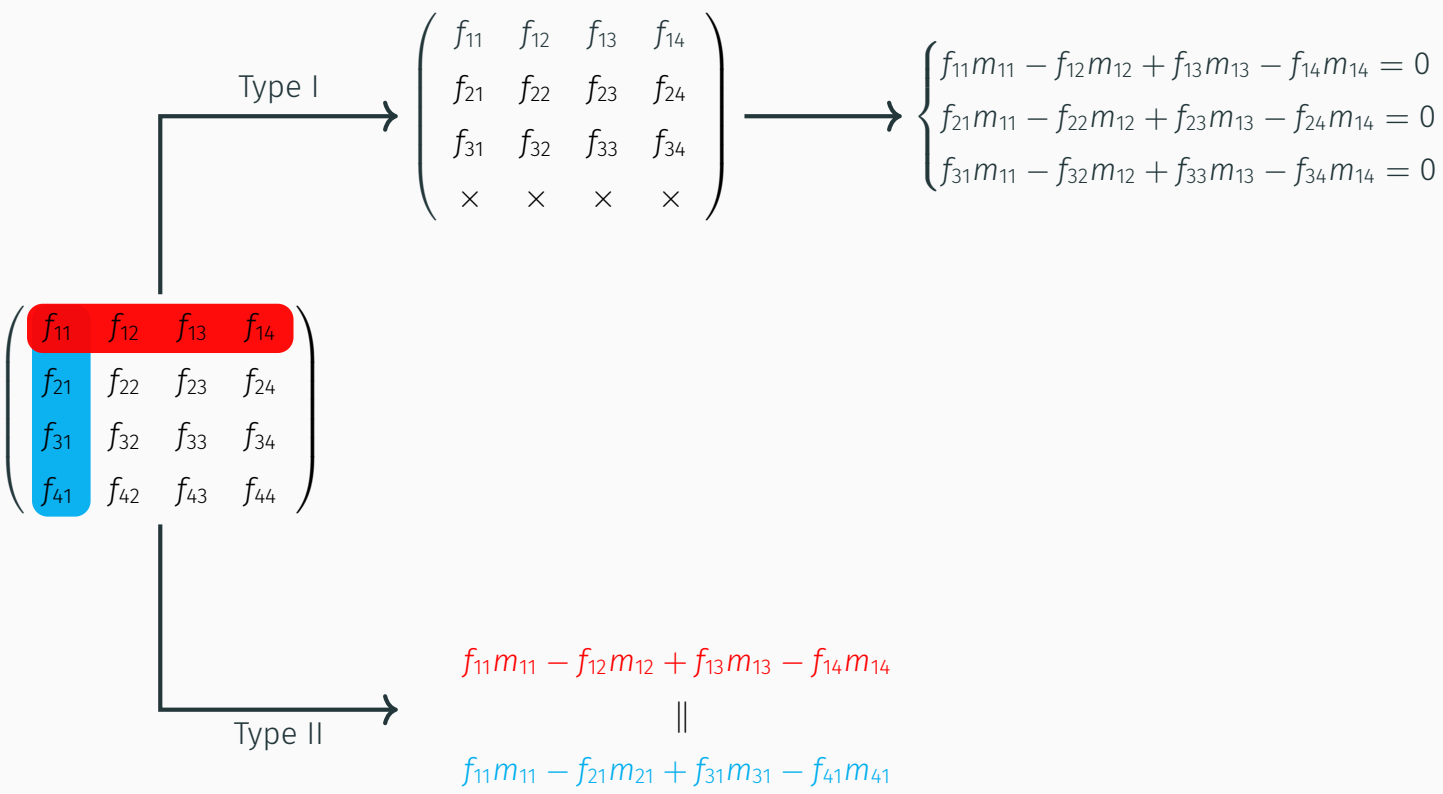

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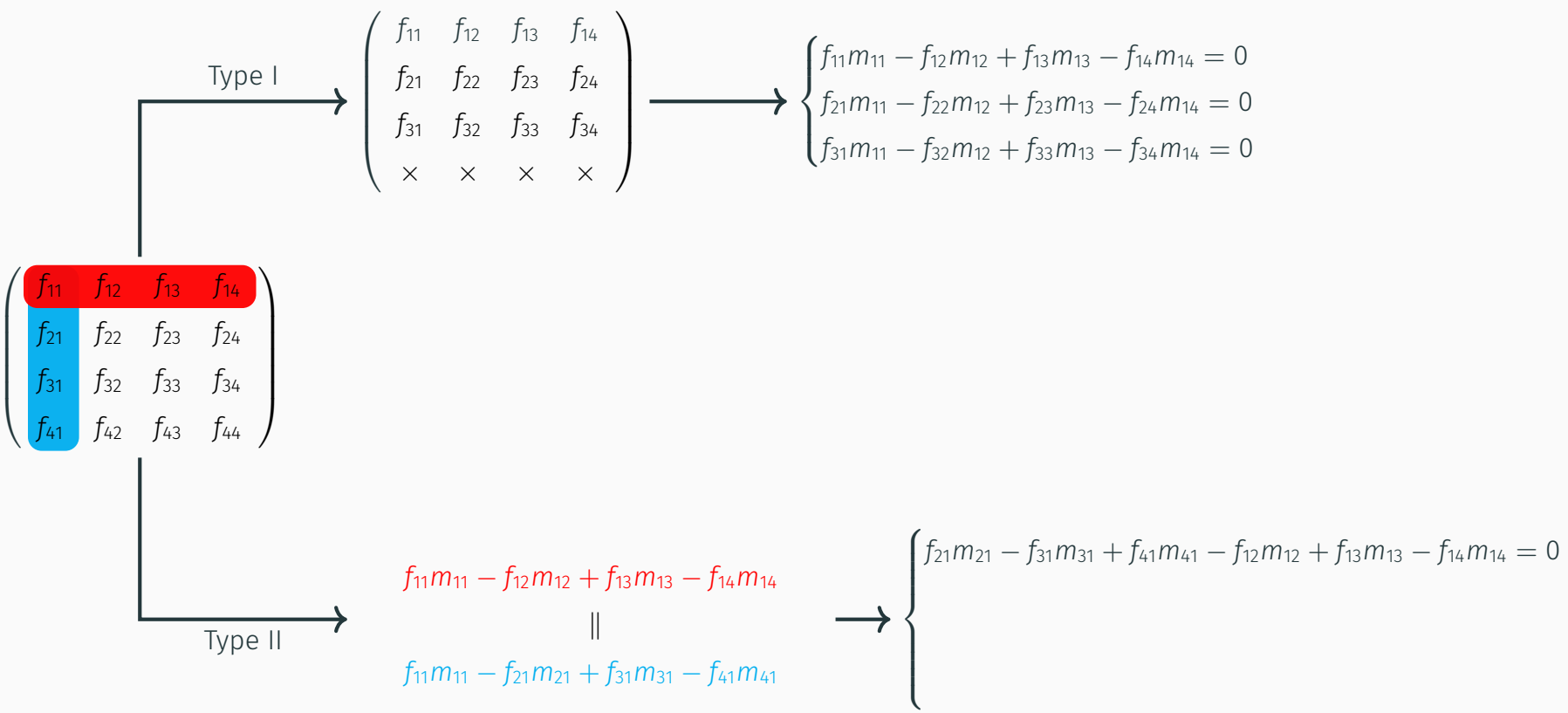

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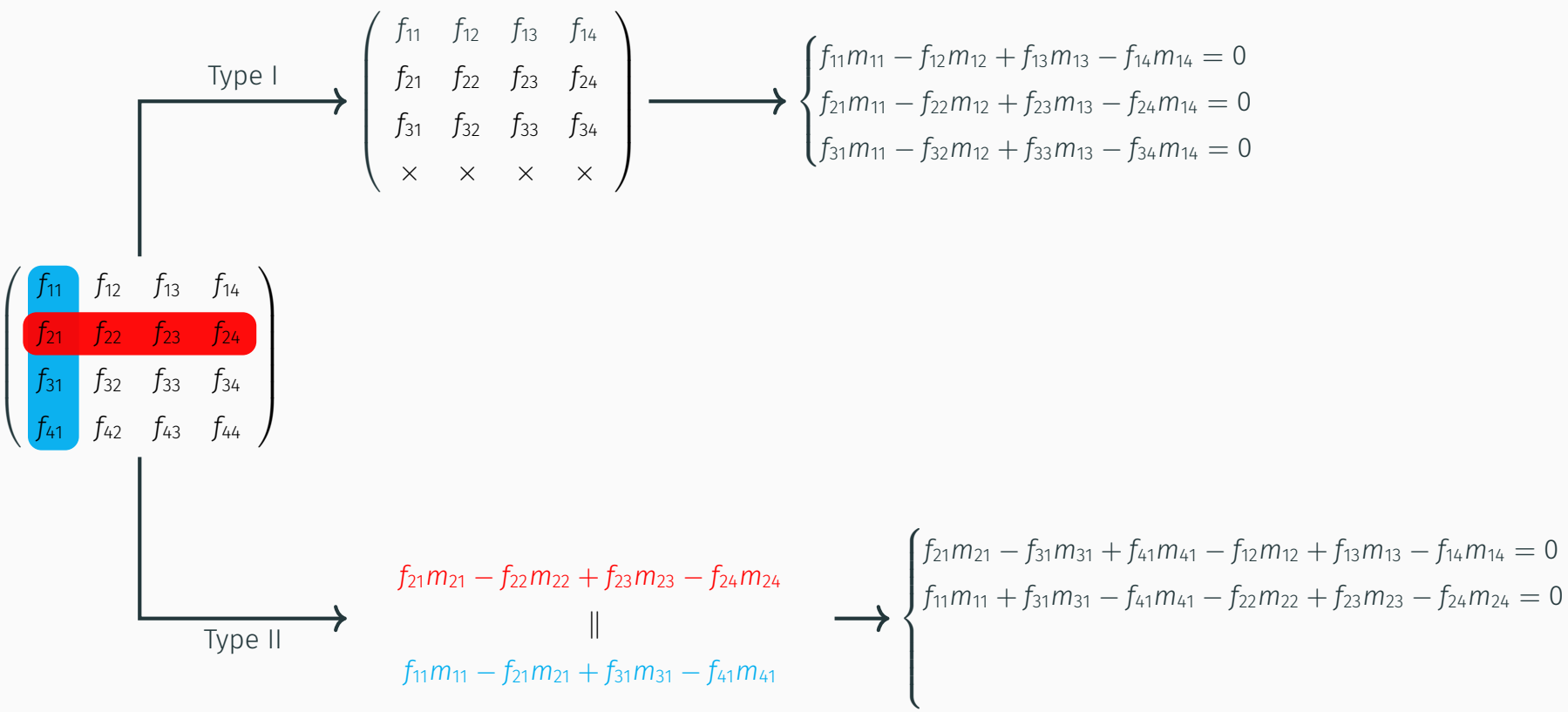

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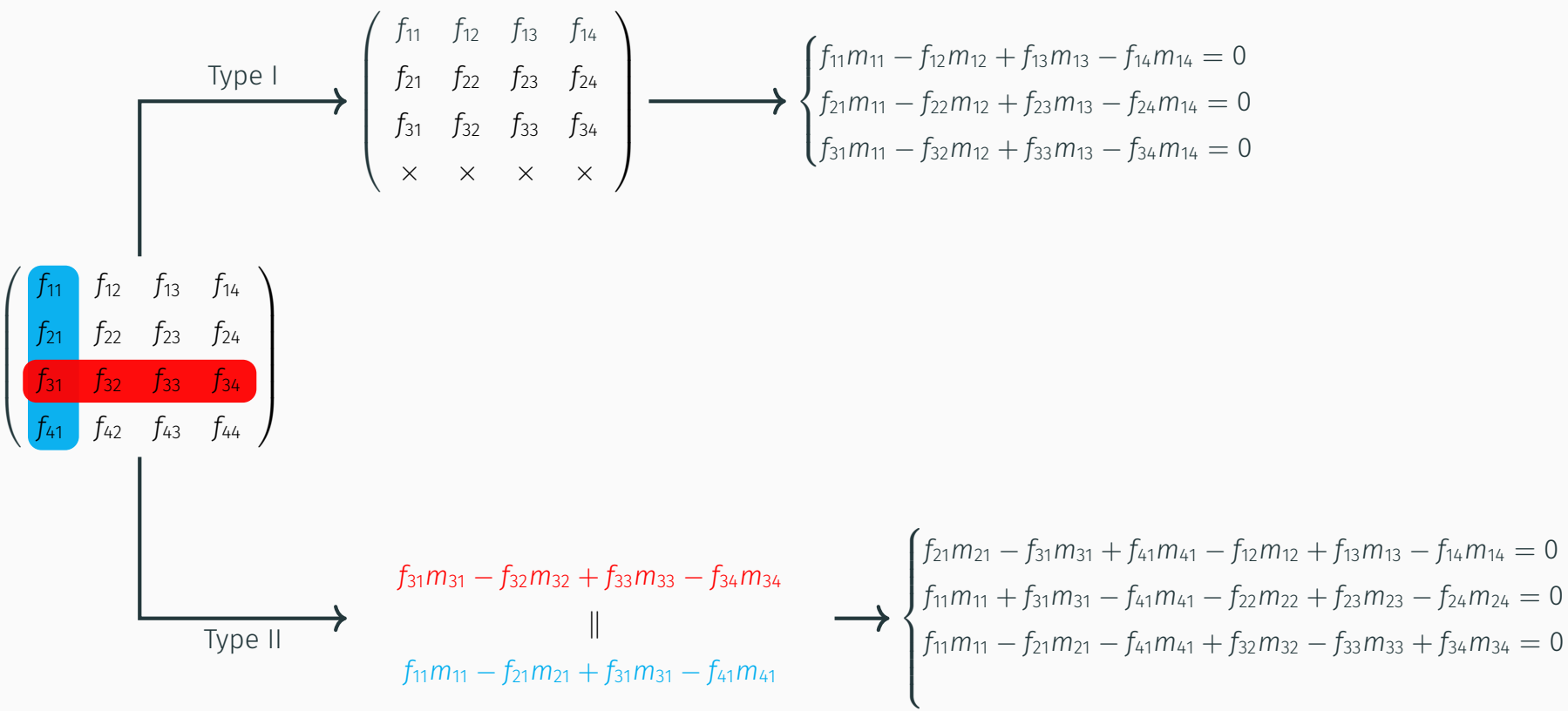

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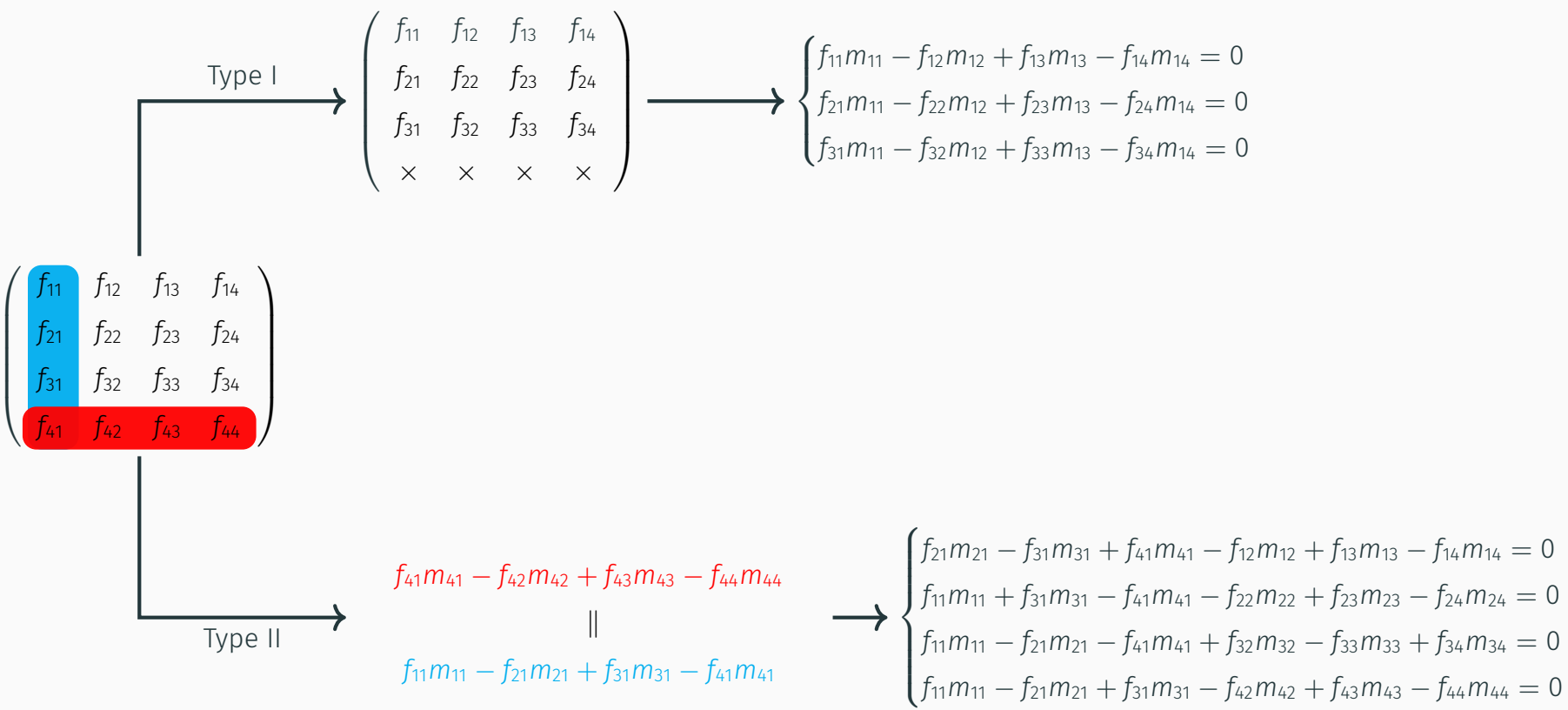

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The syzygies between the $(r+1)$-minors of M are generated by the syzygies between the $(r+1)$ minors of the $(r+2) \times (r+2)$ submatrices of M.

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Takeaway: we can reduce to the case $r = n - 2$.

New F_5 algorithms - the general case

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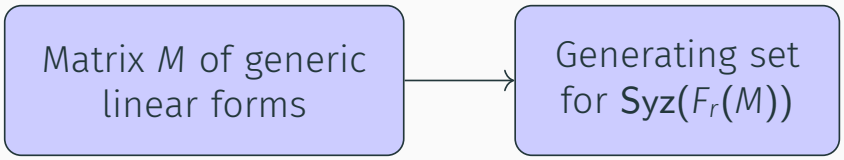
Matrix M of generic linear forms

New F_5 algorithms - the general case



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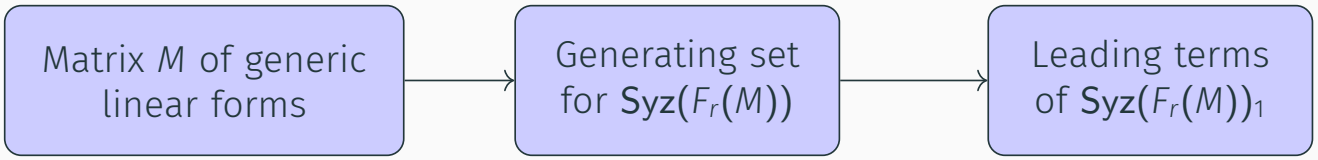

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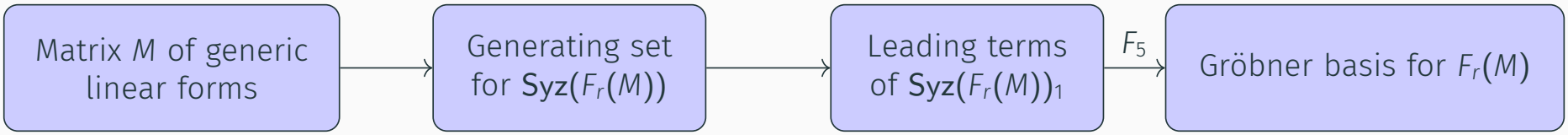

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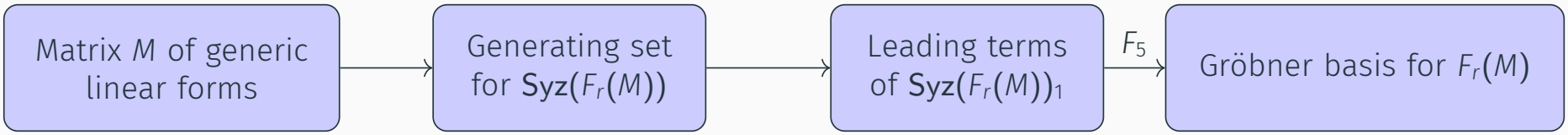

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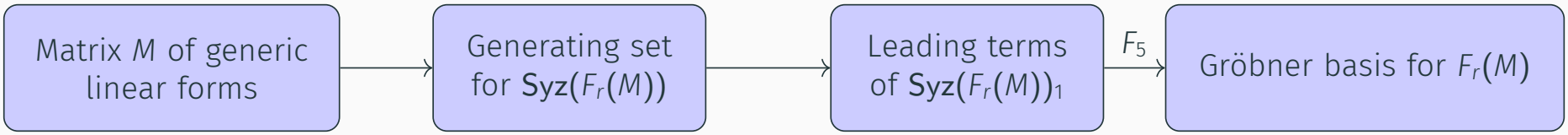

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$F_r(M)$ has a free resolution of length $(n-r)^2$.

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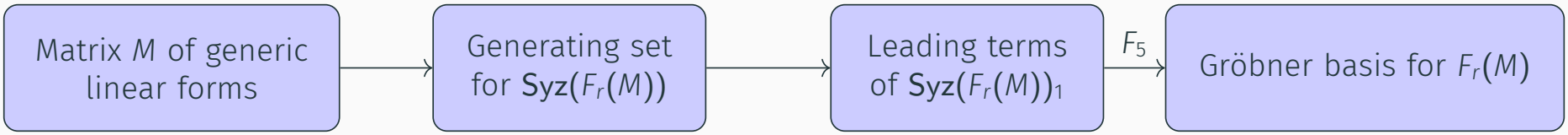

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for

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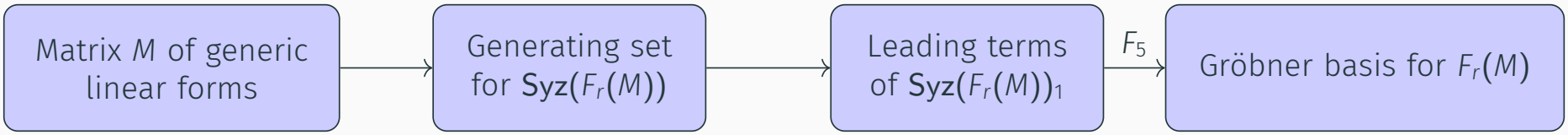

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$\mathsf{Syz}_k(F_r(M)) \neq 0$ for $1 < k < (n-r)^2$. $\Longrightarrow$ Cannot efficiently compute a Gröbner basis for $\mathsf{Syz}(F_r(M))$

New F_5 algorithms - the case $r = n - 2$

New F_5 algorithms - the case $r = n - 2$

Matrix M of generic linear forms

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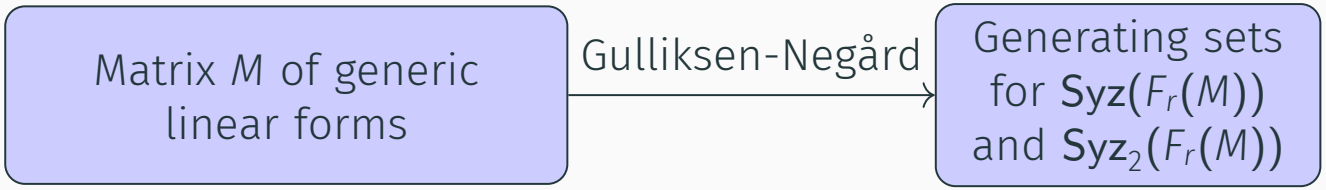

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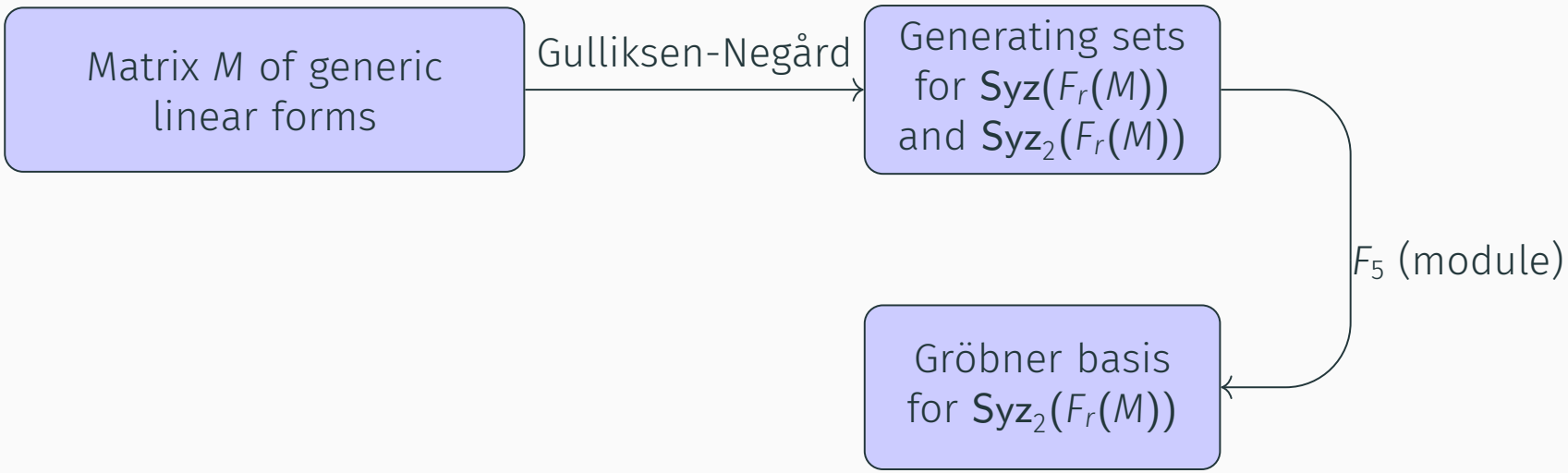

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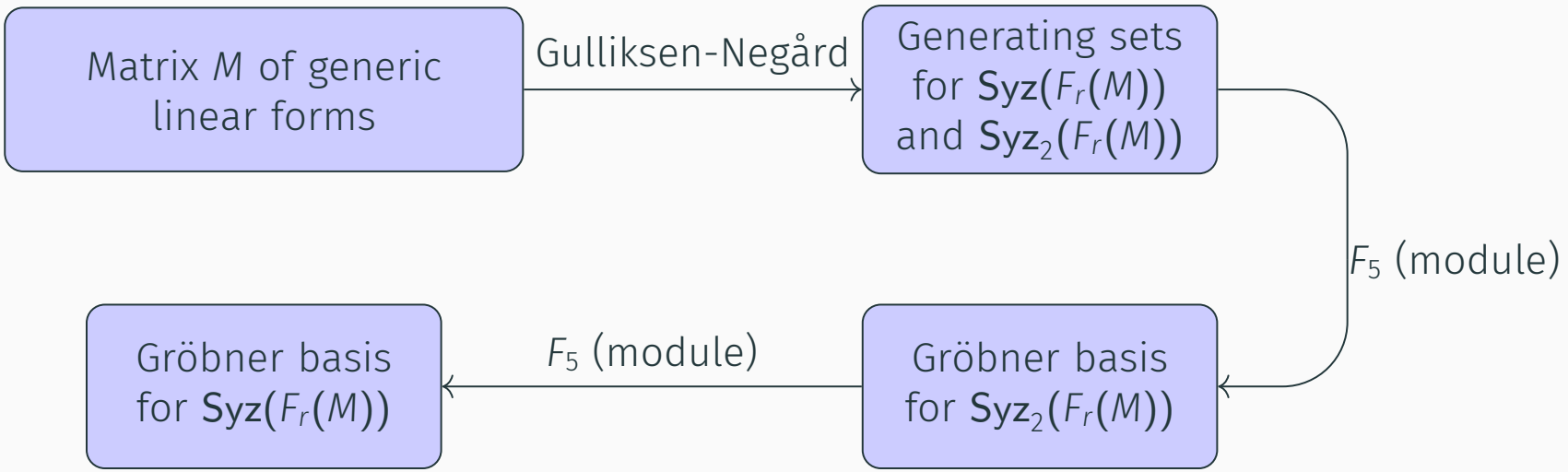

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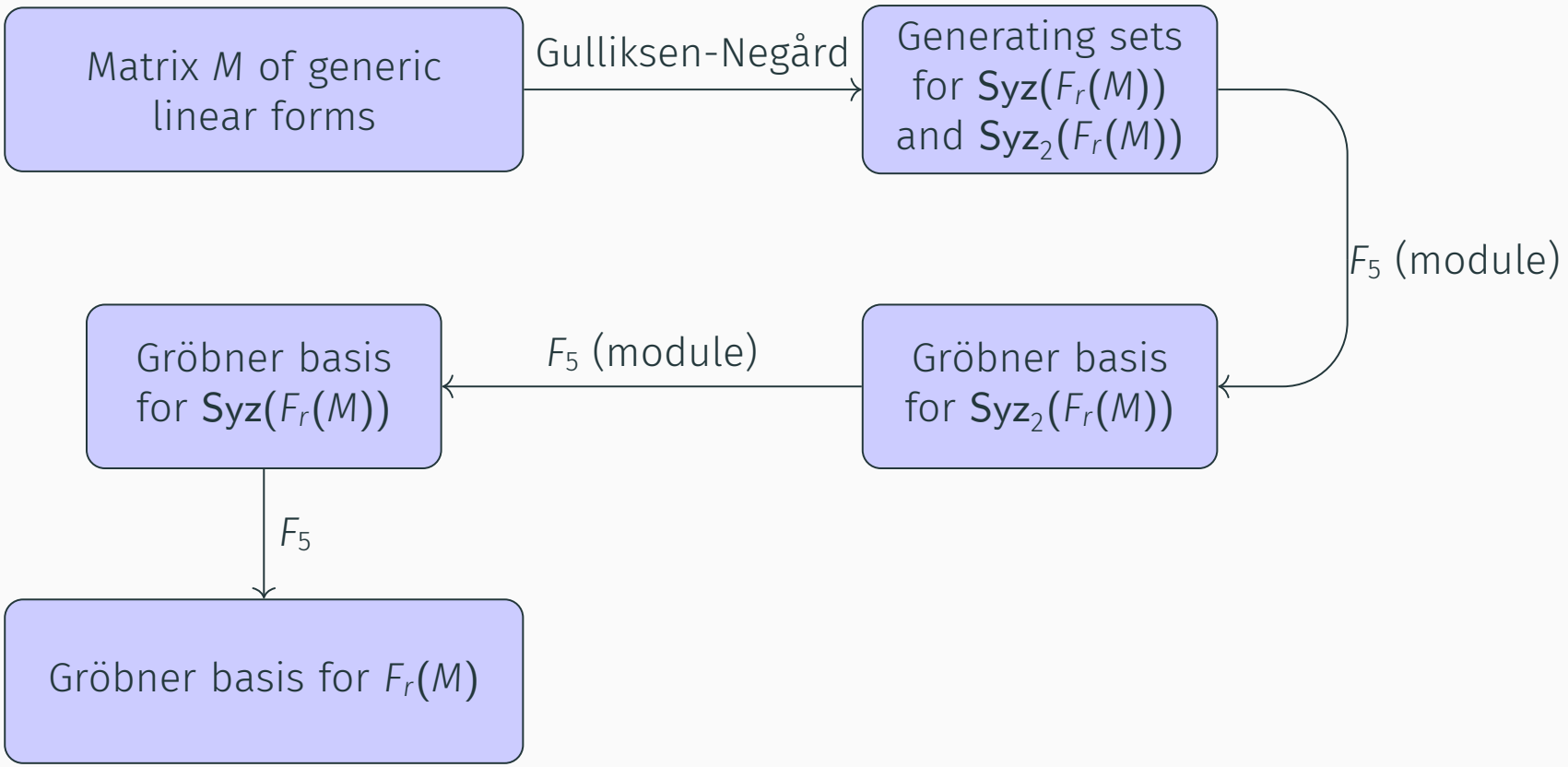

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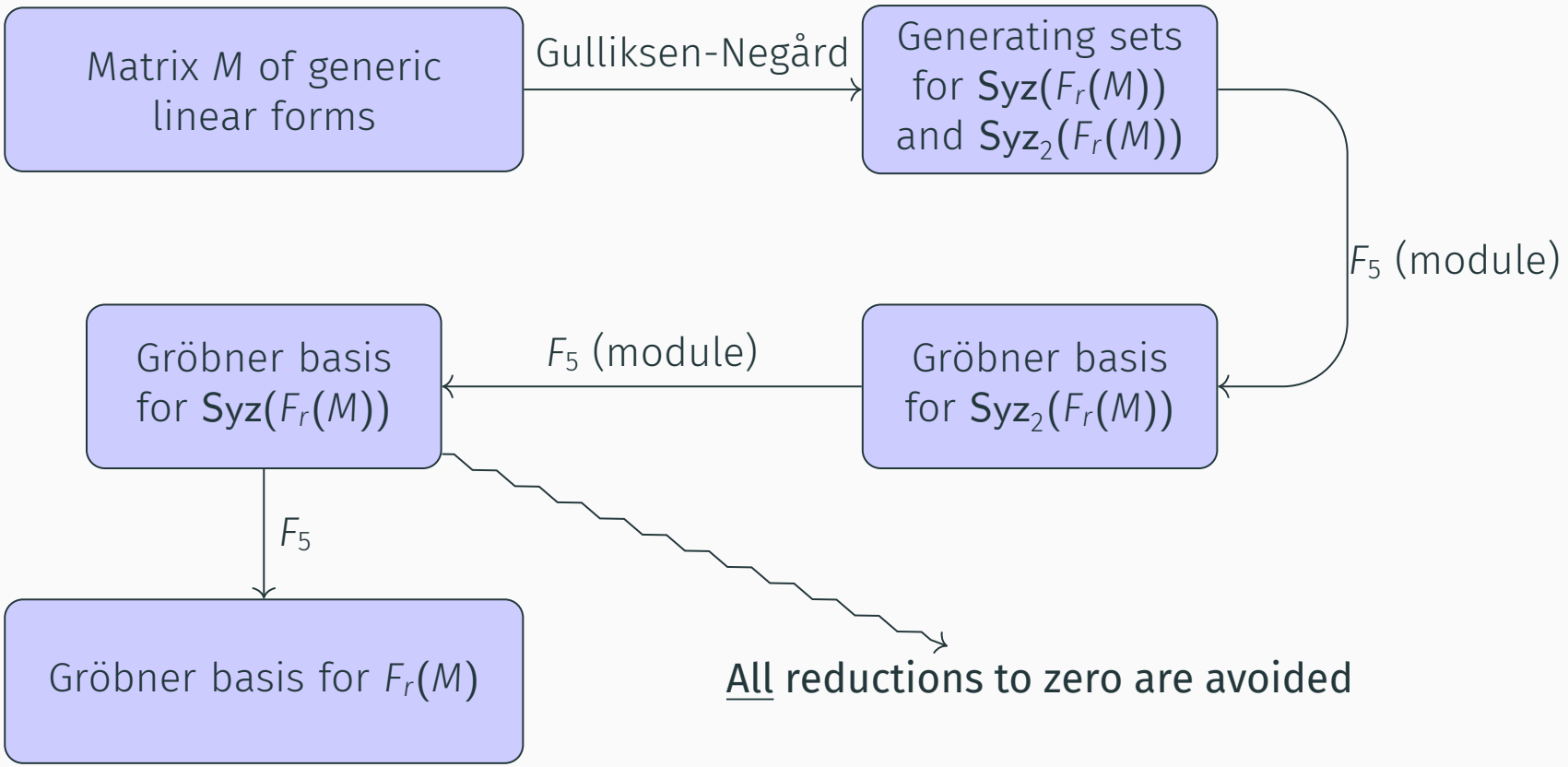

A complexity analysis in the case $r = n - 2$

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Gulliksen-
Negård complex

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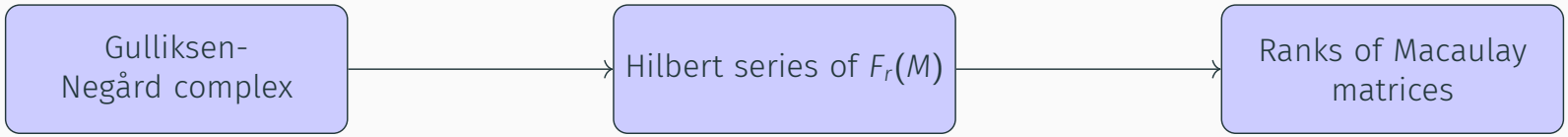

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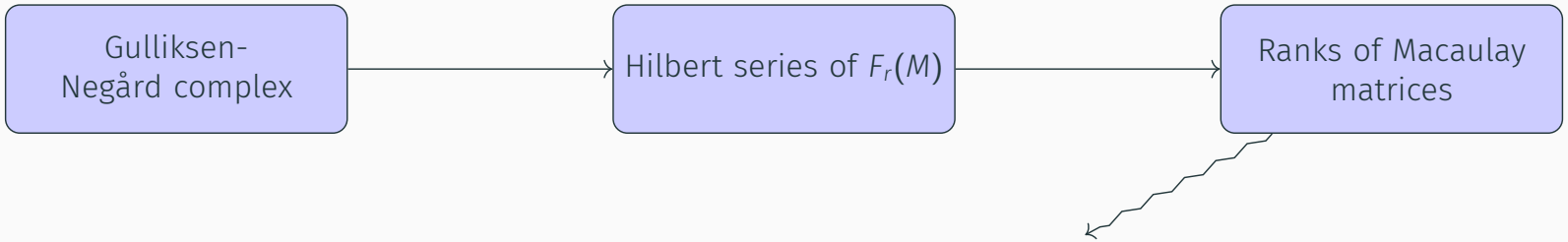

Theorem ([G., Neiger, Safey, 2023])

Let M be a matrix of generic linear forms in four variables. The complexity of computing a grevlex-Gröbner basis for $F_r(M)$ is in

$$O\left(\left(\sum_{d=n-1}^{2n-3} \frac{(d^2+(-2n+4)d+4n^2-4n+3)(2+d-n)}{3}\right)^{\omega-1}\binom{2n}{3}\right).$$

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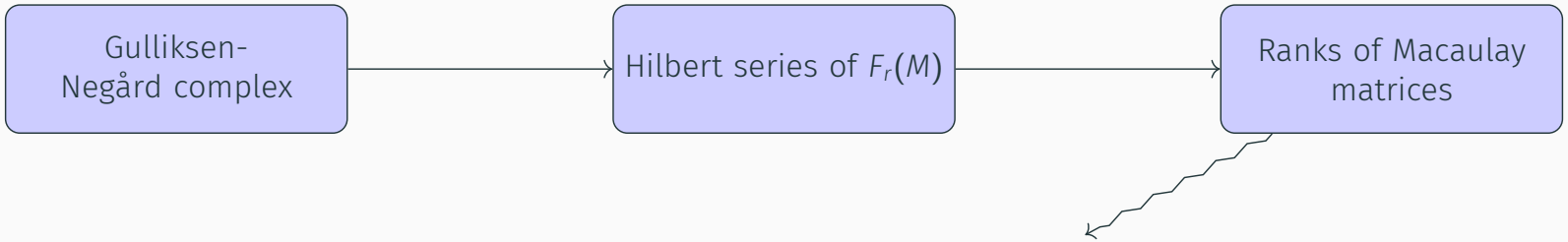

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Asymptotically:

[Faugère, Safey, Spaenlehauer, 2013]

$$O\left(n^{5\omega+2}\right)$$

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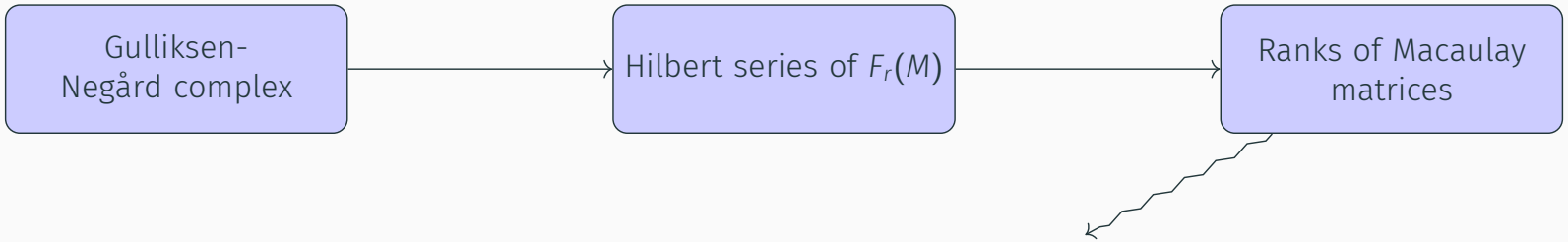

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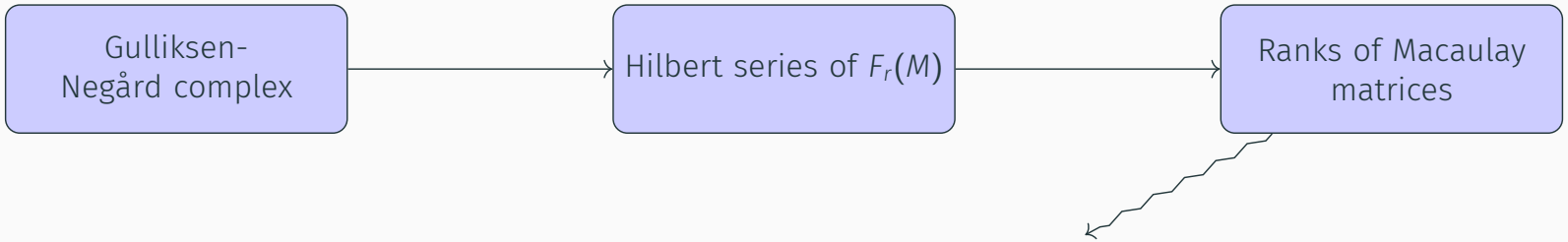

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Refined further to $O(n^{2\omega+3})$ + $|GB| \in \Omega(n^6)$...

...assuming a specific Zariski open subset of $\mathbb{A}_{\overline{k}}^{4n^2}$ is nonempty.

Hilbert series

Theorem ([Hilbert, 1890])

L graded, with ***graded*** *free resolution:*

$$0 \to \mathcal{E}_k \to \mathcal{E}_{k-1} \to \cdots \to \mathcal{E}_0 \to L \to 0.$$

Then $\mathsf{HF}_L(d) = \sum_{j=0}^{k} (-1)^j \, \mathsf{HF}_{\mathcal{E}_j}(d)$.

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Gulliksen-Negård $\rightsquigarrow$

$$\mathsf{HF}_{\mathcal{E}_0} \quad (d) =$$

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$$\mathrm{HF}_{\mathcal{E}_0(\mathrm{n}-1)}(d) = n^2 \binom{3+d-(\mathrm{n}-1)}{3}$$

$$\mathrm{HF}_{\mathcal{E}_1(\mathrm{n})}(d) = (2n^2-2) \binom{3+d-\mathrm{n}}{3}$$

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Then $\mathrm{HF}_L(d) =$

Corollary

The Macaulay matrix in degree d for $\mathcal{I}_{n-2}(M)$ *has rank*

$$n^2\binom{4+d-n}{3} - (2n^2-2)\binom{3+d-n}{3} + n^2\binom{2+d-n}{3}$$

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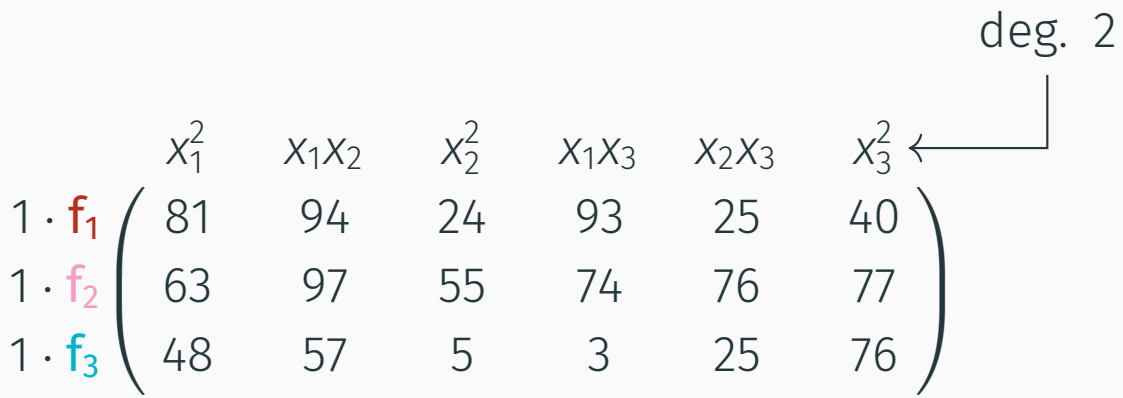
Structured Macaulay matrices

$$
\begin{array}{c}
 & \begin{array}{cccccc} x_1^2 & x_1x_2 & x_2^2 & x_1x_3 & x_2x_3 & x_3^2 \end{array} \\
\begin{array}{c} 1 \cdot f_1 \\ 1 \cdot f_2 \\ 1 \cdot f_3 \end{array} & \begin{pmatrix} 81 & 94 & 24 & 93 & 25 & 40 \\ 63 & 97 & 55 & 74 & 76 & 77 \\ 48 & 57 & 5 & 3 & 25 & 76 \end{pmatrix}
\end{array}
$$

Structured Macaulay matrices



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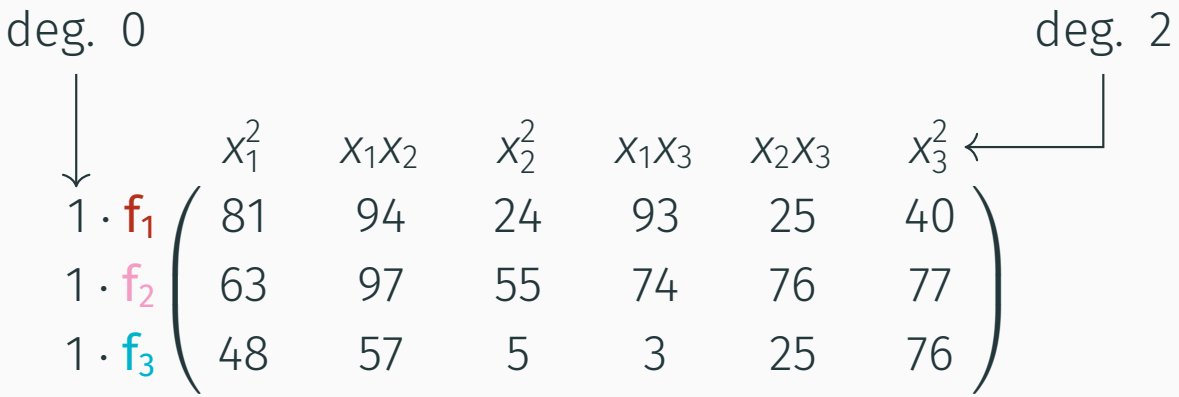
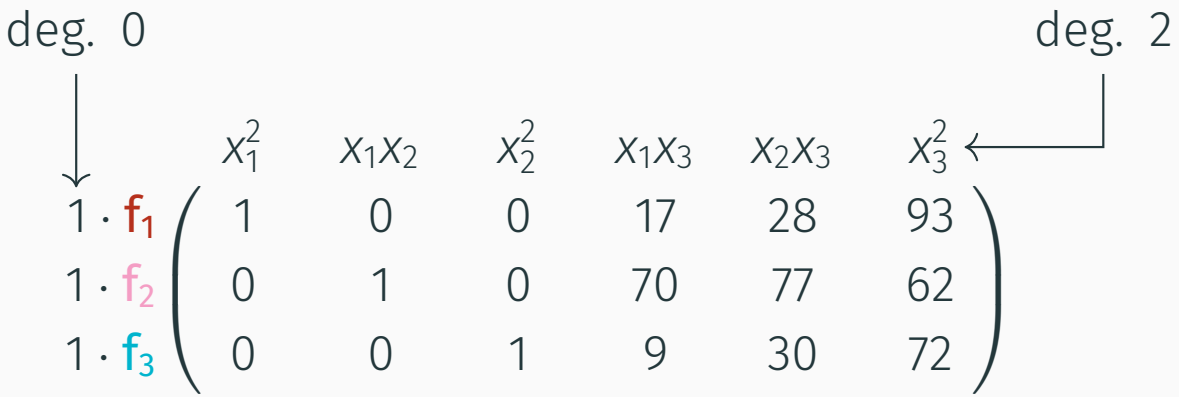


Structured Macaulay matrices



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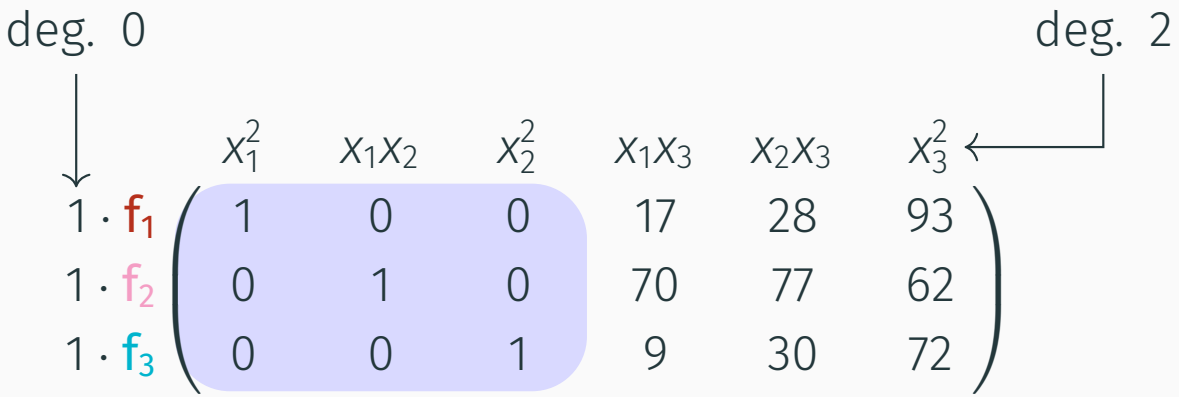

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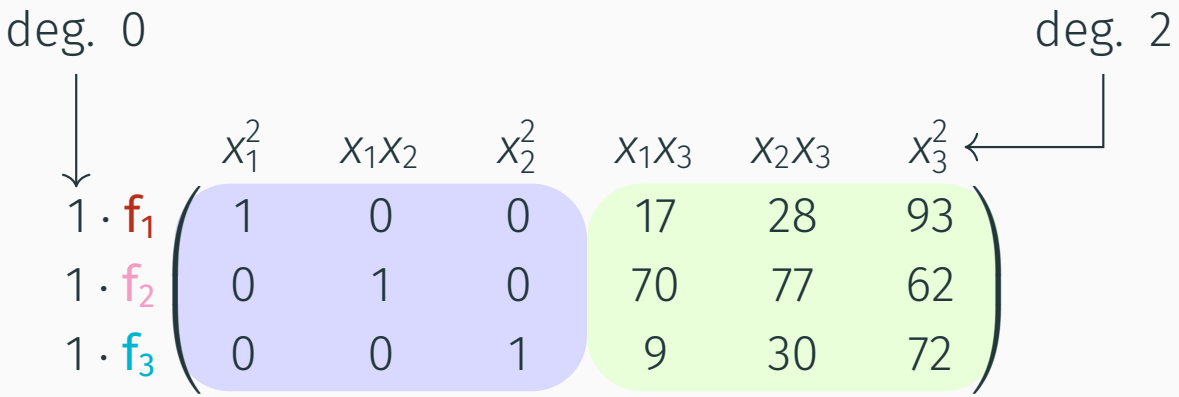
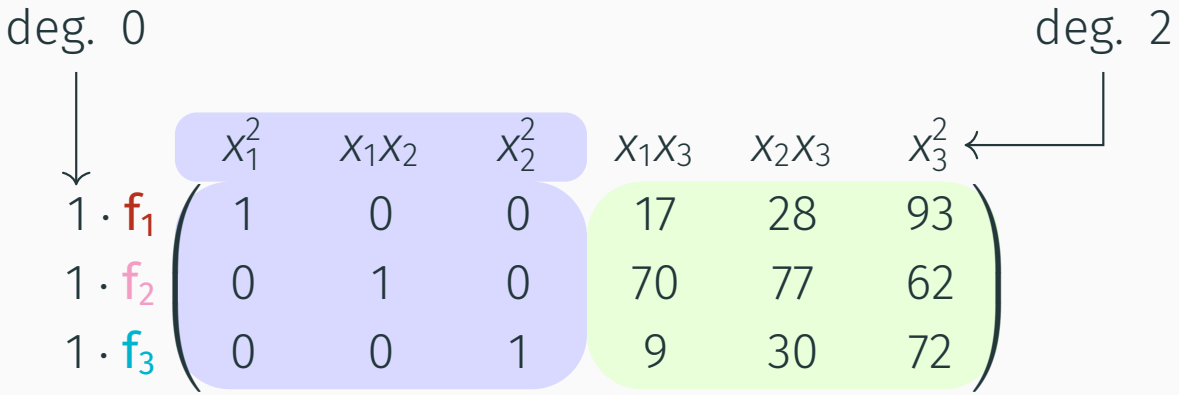


Structured Macaulay matrices

$(\mathrm{LM}_{\succ}(I))_2 = \{x_1^2, x_1x_2, x_2^2\}$



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deg. 1 ↓ deg. 3 ↓

	x_1^3	$x_1^2x_2$	$x_1x_2^2$	x_2^3	$x_1^2x_3$	$x_1x_2x_3$	$x_2^2x_3$	$x_1x_3^2$	$x_2x_3^2$	x_3^3
$x_1 \cdot f_1$	1	0	0	0	17	28	0	93	0	0
$x_2 \cdot f_1$	0	1	0	0	0	17	28	0	93	0
$x_2 \cdot f_2$	0	0	1	0	0	70	77	0	62	0
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$x_2 \cdot f_2$	0	0	1	0	0	0	0	63	38	14
$x_2 \cdot f_3$	0	0	0	1	0	0	0	9	95	9
$x_3 \cdot f_1$	0	0	0	0	1	0	0	17	28	93
$x_3 \cdot f_2$	0	0	0	0	0	1	0	70	77	62
$x_3 \cdot f_3$	0	0	0	0	0	0	1	9	30	72
$x_1 \cdot f_2$	0	0	0	0	0	0	0	75	25	68
$x_1 \cdot f_3$	0	0	0	0	0	0	0	79	26	16

Structured Macaulay matrices

$(\mathrm{LM}_{\succ}(I))_2 = \{x_1^2, x_1x_2, x_2^2\}$

deg. 1 ↓ deg. 3 ↓

	x_1^3	$x_1^2x_2$	$x_1x_2^2$	x_2^3	$x_1^2x_3$	$x_1x_2x_3$	$x_2^2x_3$	$x_1x_3^2$	$x_2x_3^2$	x_3^3
$x_1 \cdot f_1$	1	0	0	0	0	0	0	66	95	16
$x_2 \cdot f_1$	0	1	0	0	0	0	0	73	65	61
$x_2 \cdot f_2$	0	0	1	0	0	0	0	63	38	14
$x_2 \cdot f_3$	0	0	0	1	0	0	0	9	95	9
$x_3 \cdot f_1$	0	0	0	0	1	0	0	17	28	93
$x_3 \cdot f_2$	0	0	0	0	0	1	0	70	77	62
$x_3 \cdot f_3$	0	0	0	0	0	0	1	9	30	72
$x_1 \cdot f_2$	0	0	0	0	0	0	0	1	0	18
$x_1 \cdot f_3$	0	0	0	0	0	0	0	0	1	78

Structured Macaulay matrices

$(\mathrm{LM}_{\succ}(I))_2 = \{x_1^2, x_1x_2, x_2^2\}$

deg. 1 ↓ deg. 3 ↓

	x_1^3	$x_1^2x_2$	$x_1x_2^2$	x_2^3	$x_1^2x_3$	$x_1x_2x_3$	$x_2^2x_3$	$x_1x_3^2$	$x_2x_3^2$	x_3^3
$x_1 \cdot f_1$	1	0	0	0	0	0	0	66	95	16
$x_2 \cdot f_1$	0	1	0	0	0	0	0	73	65	61
$x_2 \cdot f_2$	0	0	1	0	0	0	0	63	38	14
$x_2 \cdot f_3$	0	0	0	1	0	0	0	9	95	9
$x_3 \cdot f_1$	0	0	0	0	1	0	0	17	28	93
$x_3 \cdot f_2$	0	0	0	0	0	1	0	70	77	62
$x_3 \cdot f_3$	0	0	0	0	0	0	1	9	30	72
$x_1 \cdot f_2$	0	0	0	0	0	0	0	1	0	18
$x_1 \cdot f_3$	0	0	0	0	0	0	0	0	1	78

Structured Macaulay matrices

$(\mathrm{LM}_{\succ}(I))_2 = \{x_1^2, x_1x_2, x_2^2\}$

deg. 1 ↓ deg. 3 ↓

	x_1^3	$x_1^2x_2$	$x_1x_2^2$	x_2^3	$x_1^2x_3$	$x_1x_2x_3$	$x_2^2x_3$	$x_1x_3^2$	$x_2x_3^2$	x_3^3
$x_1 \cdot f_1$	1	0	0	0	0	0	0	0	0	3
$x_2 \cdot f_1$	0	1	0	0	0	0	0	0	0	40
$x_2 \cdot f_2$	0	0	1	0	0	0	0	0	0	57
$x_2 \cdot f_3$	0	0	0	1	0	0	0	0	0	12
$x_3 \cdot f_1$	0	0	0	0	1	0	0	0	0	27
$x_3 \cdot f_2$	0	0	0	0	0	1	0	0	0	68
$x_3 \cdot f_3$	0	0	0	0	0	0	1	0	0	95
$x_1 \cdot f_2$	0	0	0	0	0	0	0	1	0	18
$x_1 \cdot f_3$	0	0	0	0	0	0	0	0	1	78

Structured Macaulay matrices

$(\mathrm{LM}_{\succ}(I))_2 = \{x_1^2, x_1x_2, x_2^2\}$

deg. 1 ↓ deg. 3 ↓

	x_1^3	$x_1^2x_2$	$x_1x_2^2$	x_2^3	$x_1^2x_3$	$x_1x_2x_3$	$x_2^2x_3$	$x_1x_3^2$	$x_2x_3^2$	x_3^3
$x_1 \cdot f_1$	1	0	0	0	0	0	0	0	0	3
$x_2 \cdot f_1$	0	1	0	0	0	0	0	0	0	40
$x_2 \cdot f_2$	0	0	1	0	0	0	0	0	0	57
$x_2 \cdot f_3$	0	0	0	1	0	0	0	0	0	12
$x_3 \cdot f_1$	0	0	0	0	1	0	0	0	0	27
$x_3 \cdot f_2$	0	0	0	0	0	1	0	0	0	68
$x_3 \cdot f_3$	0	0	0	0	0	0	1	0	0	95
$x_1 \cdot f_2$	0	0	0	0	0	0	0	1	0	18
$x_1 \cdot f_3$	0	0	0	0	0	0	0	0	1	78

Structured Macaulay matrices

$(\mathrm{LM}_\succ(I))_2 = \{x_1^2, x_1x_2, x_2^2\}$

$(\mathrm{LM}_\succ(I))_3 \setminus \langle(\mathrm{LM}_\succ(I))_2\rangle = \{x_1x_3^2, x_2x_3^2\}$

deg. 1 ↓ deg. 3 ↓

	x_1^3	$x_1^2x_2$	$x_1x_2^2$	x_2^3	$x_1^2x_3$	$x_1x_2x_3$	$x_2^2x_3$	$x_1x_3^2$	$x_2x_3^2$	x_3^3
$x_1 \cdot f_1$	1	0	0	0	0	0	0	0	0	3
$x_2 \cdot f_1$	0	1	0	0	0	0	0	0	0	40
$x_2 \cdot f_2$	0	0	1	0	0	0	0	0	0	57
$x_2 \cdot f_3$	0	0	0	1	0	0	0	0	0	12
$x_3 \cdot f_1$	0	0	0	0	1	0	0	0	0	27
$x_3 \cdot f_2$	0	0	0	0	0	1	0	0	0	68
$x_3 \cdot f_3$	0	0	0	0	0	0	1	0	0	95
$x_1 \cdot f_2$	0	0	0	0	0	0	0	1	0	18
$x_1 \cdot f_3$	0	0	0	0	0	0	0	0	1	78

Reverse lexicographic ideals

Reverse lexicographic ideals

$$
\begin{array}{c}
 & \begin{array}{ccccccc} x_1^2 & x_1x_2 & x_2^2 & x_1x_3 & x_2x_3 & x_3^2 \end{array} \\
\begin{array}{r} 1 \cdot f_1 \\ 1 \cdot f_2 \\ 1 \cdot f_3 \end{array} & \left(\begin{array}{ccccccc} 1 & 0 & 0 & 17 & 28 & 93 \\ 0 & 1 & 0 & 70 & 77 & 62 \\ 0 & 0 & 1 & 9 & 30 & 72 \end{array} \right)
\end{array}
$$

Reverse lexicographic ideals

contiguous block

	x_1^2	x_1x_2	x_2^2	x_1x_3	x_2x_3	x_3^2
$1 \cdot f_1$	1	0	0	17	28	93
$1 \cdot f_2$	0	1	0	70	77	62
$1 \cdot f_3$	0	0	1	9	30	72

Reverse lexicographic ideals

	x_1^3	$x_1^2x_2$	$x_1x_2^2$	x_2^3	$x_1^2x_3$	$x_1x_2x_3$	$x_2^2x_3$	$x_1x_3^2$	$x_2x_3^2$	x_3^3
$x_1 \cdot f_1$	1	0	0	0	0	0	0	0	0	3
$x_2 \cdot f_1$	0	1	0	0	0	0	0	0	0	40
$x_2 \cdot f_2$	0	0	1	0	0	0	0	0	0	57
$x_2 \cdot f_3$	0	0	0	1	0	0	0	0	0	12
$x_3 \cdot f_1$	0	0	0	0	1	0	0	0	0	27
$x_3 \cdot f_2$	0	0	0	0	0	1	0	0	0	68
$x_3 \cdot f_3$	0	0	0	0	0	0	1	0	0	95
$x_1 \cdot f_2$	0	0	0	0	0	0	0	1	0	18
$x_1 \cdot f_3$	0	0	0	0	0	0	0	0	1	78

Reverse lexicographic ideals

	x_1^3	$x_1^2x_2$	$x_1x_2^2$	x_2^3	$x_1^2x_3$	$x_1x_2x_3$	$x_2^2x_3$	$x_1x_3^2$	$x_2x_3^2$	x_3^3
$x_1 \cdot f_1$	1	0	0	0	0	0	0	0	0	3
$x_2 \cdot f_1$	0	1	0	0	0	0	0	0	0	40
$x_2 \cdot f_2$	0	0	1	0	0	0	0	0	0	57
$x_2 \cdot f_3$	0	0	0	1	0	0	0	0	0	12
$x_3 \cdot f_1$	0	0	0	0	1	0	0	0	0	27
$x_3 \cdot f_2$	0	0	0	0	0	1	0	0	0	68
$x_3 \cdot f_3$	0	0	0	0	0	0	1	0	0	95
$x_1 \cdot f_2$	0	0	0	0	0	0	0	1	0	18
$x_1 \cdot f_3$	0	0	0	0	0	0	0	0	1	78

Reverse lexicographic ideals

contiguous block

	x_1^3	$x_1^2x_2$	$x_1x_2^2$	x_2^3	$x_1^2x_3$	$x_1x_2x_3$	$x_2^2x_3$	$x_1x_3^2$	$x_2x_3^2$	x_3^3
$x_1 \cdot f_1$	1	0	0	0	0	0	0	0	0	3
$x_2 \cdot f_1$	0	1	0	0	0	0	0	0	0	40
$x_2 \cdot f_2$	0	0	1	0	0	0	0	0	0	57
$x_2 \cdot f_3$	0	0	0	1	0	0	0	0	0	12
$x_3 \cdot f_1$	0	0	0	0	1	0	0	0	0	27
$x_3 \cdot f_2$	0	0	0	0	0	1	0	0	0	68
$x_3 \cdot f_3$	0	0	0	0	0	0	1	0	0	95
$x_1 \cdot f_2$	0	0	0	0	0	0	0	1	0	18
$x_1 \cdot f_3$	0	0	0	0	0	0	0	0	1	78

Reverse lexicographic ideals

contiguous block

	x_1^3	$x_1^2x_2$	$x_1x_2^2$	x_2^3	$x_1^2x_3$	$x_1x_2x_3$	$x_2^2x_3$	$x_1x_3^2$	$x_2x_3^2$	x_3^3
$x_1 \cdot f_1$	1	0	0	0	0	0	0	0	0	3
$x_2 \cdot f_1$	0	1	0	0	0	0	0	0	0	40
$x_2 \cdot f_2$	0	0	1	0	0	0	0	0	0	57
$x_2 \cdot f_3$	0	0	0	1	0	0	0	0	0	12
$x_3 \cdot f_1$	0	0	0	0	1	0	0	0	0	27
$x_3 \cdot f_2$	0	0	0	0	0	1	0	0	0	68
$x_3 \cdot f_3$	0	0	0	0	0	0	1	0	0	95
$x_1 \cdot f_2$	0	0	0	0	0	0	0	1	0	18
$x_1 \cdot f_3$	0	0	0	0	0	0	0	0	1	78

Definition (Reverse lexicographic ideal)

An ideal $\mathcal{I}$ is called **reverse lexicographic** if for any $\sigma \in \mathbf{LM}_{\succ}(\mathcal{I})$ and for any monomial τ of degree $\deg(\sigma)$, if $\tau \succ_{drl} \sigma$, then $\tau \in \mathbf{LM}_{\succ}(\mathcal{I})$.

Reverse lexicographic ideals

contiguous block

	x_1^3	$x_1^2x_2$	$x_1x_2^2$	x_2^3	$x_1^2x_3$	$x_1x_2x_3$	$x_2^2x_3$	$x_1x_3^2$	$x_2x_3^2$	x_3^3
$x_1 \cdot f_1$	1	0	0	0	0	0	0	0	0	3
$x_2 \cdot f_1$	0	1	0	0	0	0	0	0	0	40
$x_2 \cdot f_2$	0	0	1	0	0	0	0	0	0	57
$x_2 \cdot f_3$	0	0	0	1	0	0	0	0	0	12
$x_3 \cdot f_1$	0	0	0	0	1	0	0	0	0	27
$x_3 \cdot f_2$	0	0	0	0	0	1	0	0	0	68
$x_3 \cdot f_3$	0	0	0	0	0	0	1	0	0	95
$x_1 \cdot f_2$	0	0	0	0	0	0	0	1	0	18
$x_1 \cdot f_3$	0	0	0	0	0	0	0	0	1	78

Definition (Reverse lexicographic ideal)

An ideal $\mathcal{I}$ is called **reverse lexicographic** if for any $\sigma \in \mathbf{LM}_{\succ}(\mathcal{I})$ and for any monomial τ of degree $\deg(\sigma)$, if $\tau \succ_{drl} \sigma$, then $\tau \in \mathbf{LM}_{\succ}(\mathcal{I})$.

Reverse lexicographic ideals

contiguous block

	x_1^3	$x_1^2x_2$	$x_1x_2^2$	x_2^3	$x_1^2x_3$	$x_1x_2x_3$	$x_2^2x_3$	$x_1x_3^2$	$x_2x_3^2$	x_3^3
$x_1 \cdot f_1$	1	0	0	0	0	0	0	0	0	3
$x_2 \cdot f_1$	0	1	0	0	0	0	0	0	0	40
$x_2 \cdot f_2$	0	0	1	0	0	0	0	0	0	57
$x_2 \cdot f_3$	0	0	0	1	0	0	0	0	0	12
$x_3 \cdot f_1$	0	0	0	0	1	0	0	0	0	27
$x_3 \cdot f_2$	0	0	0	0	0	1	0	0	0	68
$x_3 \cdot f_3$	0	0	0	0	0	0	1	0	0	95
$x_1 \cdot f_2$	0	0	0	0	0	0	0	1	0	18
$x_1 \cdot f_3$	0	0	0	0	0	0	0	0	1	78

Theorem

If for any degree d, the reduced row echelon form of the Macaulay matrix $\mathcal{M}_d$ is of the form $\left(\begin{array}{c|c} I & X \end{array} \right)$, then $\mathcal{I}$ is reverse lexicographic.

Reverse lexicographic ideals

contiguous block

	x_1^3	$x_1^2x_2$	$x_1x_2^2$	x_2^3	$x_1^2x_3$	$x_1x_2x_3$	$x_2^2x_3$	$x_1x_3^2$	$x_2x_3^2$	x_3^3
$x_1 \cdot f_1$	1	0	0	0	0	0	0	0	0	3
$x_2 \cdot f_1$	0	1	0	0	0	0	0	0	0	40
$x_2 \cdot f_2$	0	0	1	0	0	0	0	0	0	57
$x_2 \cdot f_3$	0	0	0	1	0	0	0	0	0	12
$x_3 \cdot f_1$	0	0	0	0	1	0	0	0	0	27
$x_3 \cdot f_2$	0	0	0	0	0	1	0	0	0	68
$x_3 \cdot f_3$	0	0	0	0	0	0	1	0	0	95
$x_1 \cdot f_2$	0	0	0	0	0	0	0	1	0	18
$x_1 \cdot f_3$	0	0	0	0	0	0	0	0	1	78

Genericity: $\det(\quad) \neq 0$.

Theorem

If for any degree d, the reduced row echelon form of the Macaulay matrix $\mathcal{M}_d$ *is of the form* $\left(\begin{array}{c|c} I & X \end{array} \right)$, *then* $\mathcal{I}$ *is reverse lexicographic.*

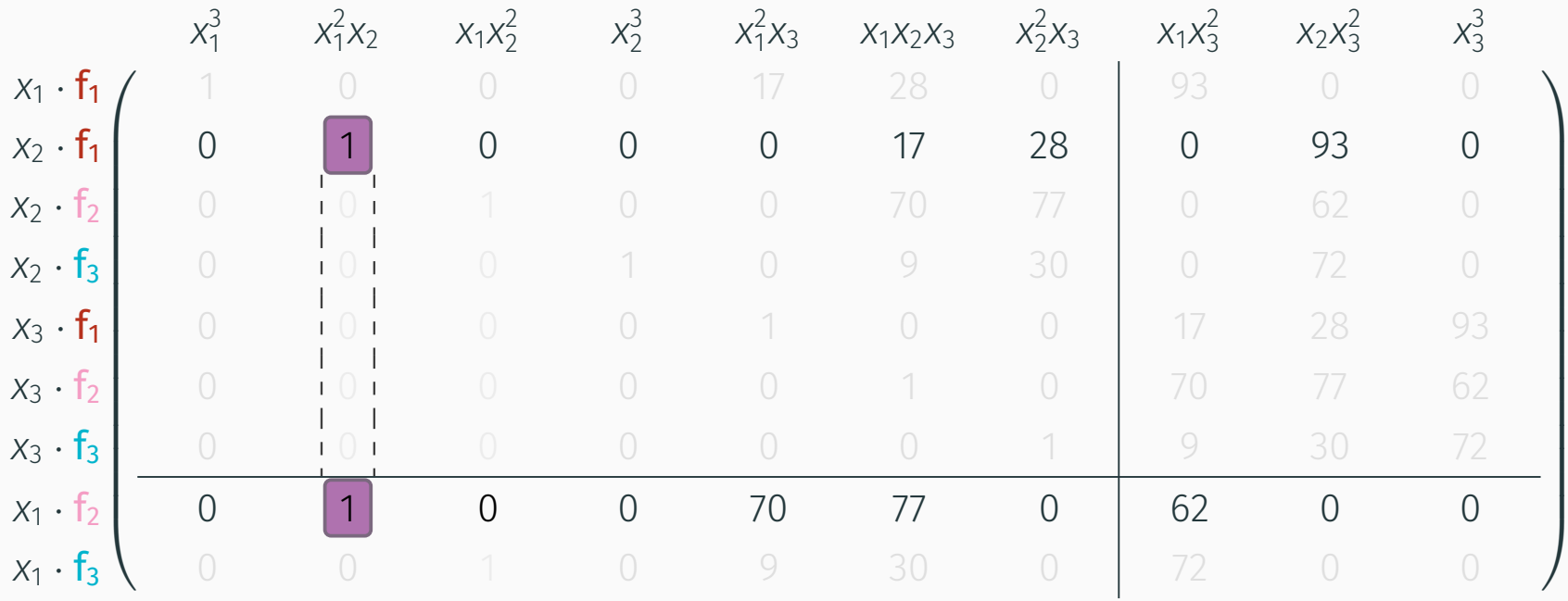
Bounds on the size of the reduced grevlex Gröbner basis

	x_1^3	$x_1^2x_2$	$x_1x_2^2$	x_2^3	$x_1^2x_3$	$x_1x_2x_3$	$x_2^2x_3$	$x_1x_3^2$	$x_2x_3^2$	x_3^3
$x_1 \cdot f_1$	1	0	0	0	17	28	0	93	0	0
$x_2 \cdot f_1$	0	1	0	0	0	17	28	0	93	0
$x_2 \cdot f_2$	0	0	1	0	0	70	77	0	62	0
$x_2 \cdot f_3$	0	0	0	1	0	9	30	0	72	0
$x_3 \cdot f_1$	0	0	0	0	1	0	0	17	28	93
$x_3 \cdot f_2$	0	0	0	0	0	1	0	70	77	62
$x_3 \cdot f_3$	0	0	0	0	0	0	1	9	30	72
$x_1 \cdot f_2$	0	1	0	0	70	77	0	62	0	0
$x_1 \cdot f_3$	0	0	1	0	9	30	0	72	0	0

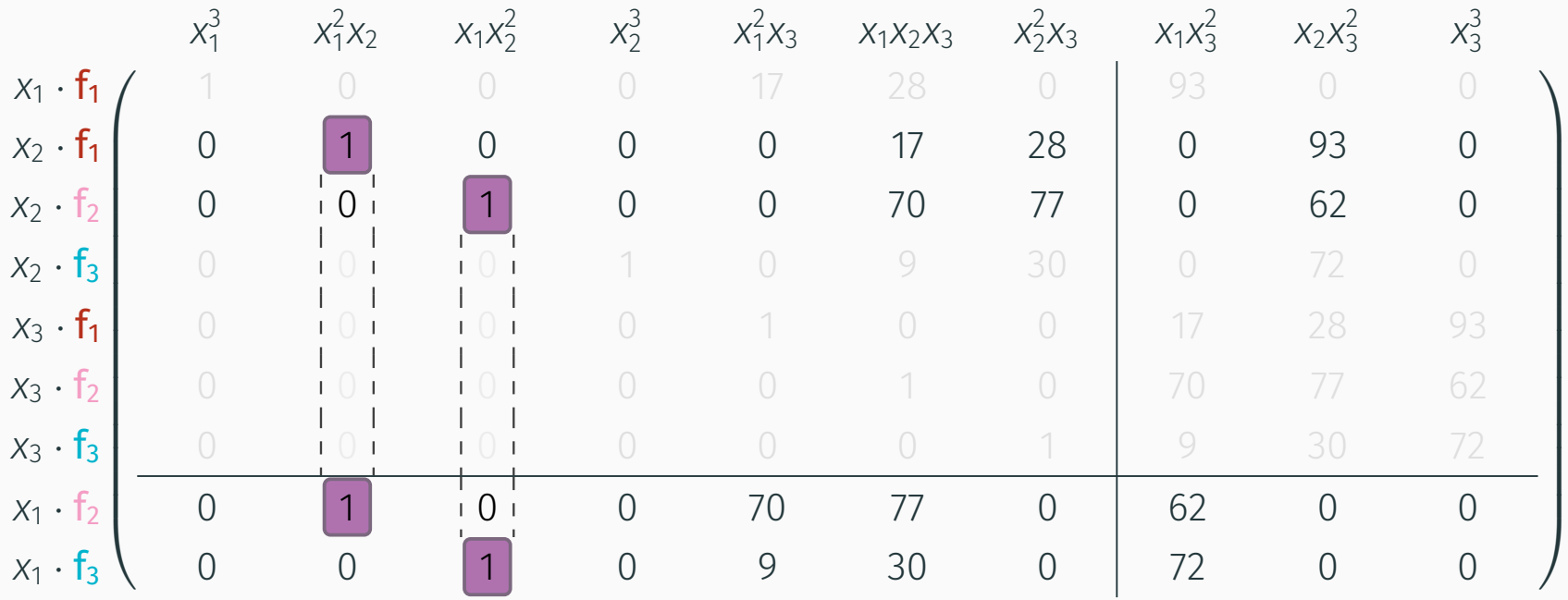
Bounds on the size of the reduced grevlex Gröbner basis

	x_1^3	$x_1^2x_2$	$x_1x_2^2$	x_2^3	$x_1^2x_3$	$x_1x_2x_3$	$x_2^2x_3$	$x_1x_3^2$	$x_2x_3^2$	x_3^3
$x_1 \cdot f_1$	1	0	0	0	17	28	0	93	0	0
$x_2 \cdot f_1$	0	1	0	0	0	17	28	0	93	0
$x_2 \cdot f_2$	0	0	1	0	0	70	77	0	62	0
$x_2 \cdot f_3$	0	0	0	1	0	9	30	0	72	0
$x_3 \cdot f_1$	0	0	0	0	1	0	0	17	28	93
$x_3 \cdot f_2$	0	0	0	0	0	1	0	70	77	62
$x_3 \cdot f_3$	0	0	0	0	0	0	1	9	30	72
$x_1 \cdot f_2$	0	1	0	0	70	77	0	62	0	0
$x_1 \cdot f_3$	0	0	1	0	9	30	0	72	0	0

Bounds on the size of the reduced grevlex Gröbner basis



Bounds on the size of the reduced grevlex Gröbner basis



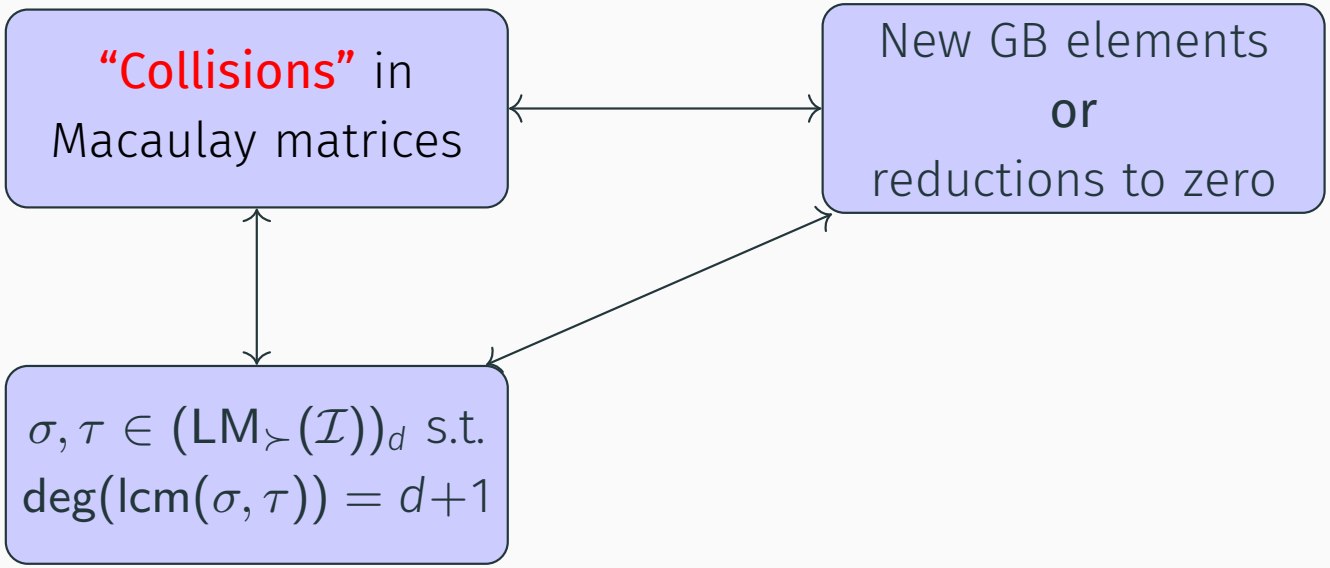
Bounds on the size of the reduced grevlex Gröbner basis

	x_1^3	$x_1^2x_2$	$x_1x_2^2$	x_2^3	$x_1^2x_3$	$x_1x_2x_3$	$x_2^2x_3$	$x_1x_3^2$	$x_2x_3^2$	x_3^3
$x_1 \cdot f_1$	1	0	0	0	17	28	0	93	0	0
$x_2 \cdot f_1$	0	1	0	0	0	17	28	0	93	0
$x_2 \cdot f_2$	0	0	1	0	0	70	77	0	62	0
$x_2 \cdot f_3$	0	0	0	1	0	9	30	0	72	0
$x_3 \cdot f_1$	0	0	0	0	1	0	0	17	28	93
$x_3 \cdot f_2$	0	0	0	0	0	1	0	70	77	62
$x_3 \cdot f_3$	0	0	0	0	0	0	1	9	30	72
$x_1 \cdot f_2$	0	1	0	0	70	77	0	62	0	0
$x_1 \cdot f_3$	0	0	1	0	9	30	0	72	0	0

"Collisions" in Macaulay matrices ⟷ New GB elements **or** reductions to zero

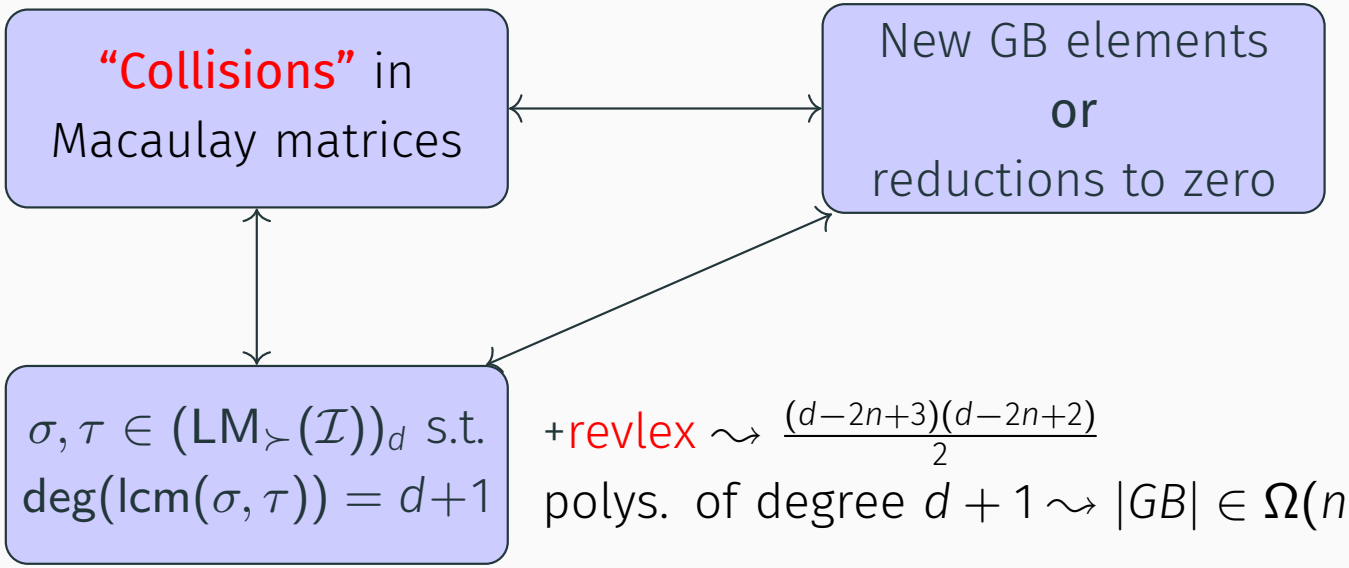
Bounds on the size of the reduced grevlex Gröbner basis

	x_1^3	$x_1^2x_2$	$x_1x_2^2$	x_2^3	$x_1^2x_3$	$x_1x_2x_3$	$x_2^2x_3$	$x_1x_3^2$	$x_2x_3^2$	x_3^3
$x_1 \cdot f_1$	1	0	0	0	17	28	0	93	0	0
$x_2 \cdot f_1$	0	1	0	0	0	17	28	0	93	0
$x_2 \cdot f_2$	0	0	1	0	0	70	77	0	62	0
$x_2 \cdot f_3$	0	0	0	1	0	9	30	0	72	0
$x_3 \cdot f_1$	0	0	0	0	1	0	0	17	28	93
$x_3 \cdot f_2$	0	0	0	0	0	1	0	70	77	62
$x_3 \cdot f_3$	0	0	0	0	0	0	1	9	30	72
$x_1 \cdot f_2$	0	1	0	0	70	77	0	62	0	0
$x_1 \cdot f_3$	0	0	1	0	9	30	0	72	0	0



Bounds on the size of the reduced grevlex Gröbner basis

	x_1^3	$x_1^2x_2$	$x_1x_2^2$	x_2^3	$x_1^2x_3$	$x_1x_2x_3$	$x_2^2x_3$	$x_1x_3^2$	$x_2x_3^2$	x_3^3
$x_1 \cdot f_1$	1	0	0	0	17	28	0	93	0	0
$x_2 \cdot f_1$	0	1	0	0	0	17	28	0	93	0
$x_2 \cdot f_2$	0	0	1	0	0	70	77	0	62	0
$x_2 \cdot f_3$	0	0	0	1	0	9	30	0	72	0
$x_3 \cdot f_1$	0	0	0	0	1	0	0	17	28	93
$x_3 \cdot f_2$	0	0	0	0	0	1	0	70	77	62
$x_3 \cdot f_3$	0	0	0	0	0	0	1	9	30	72
$x_1 \cdot f_2$	0	1	0	0	70	77	0	62	0	0
$x_1 \cdot f_3$	0	0	1	0	9	30	0	72	0	0



Upper bounds

	x_1^3	$x_1^2x_2$	$x_1x_2^2$	x_2^3	$x_1^2x_3$	$x_1x_2x_3$	$x_2^2x_3$	$x_1x_3^2$	$x_2x_3^2$	x_3^3
$x_1 \cdot f_1$	1	0	0	0	17	28	0	93	0	0
$x_2 \cdot f_1$	0	1	0	0	0	17	28	0	93	0
$x_2 \cdot f_2$	0	0	1	0	0	70	77	0	62	0
$x_2 \cdot f_3$	0	0	0	1	0	9	30	0	72	0
$x_3 \cdot f_1$	0	0	0	0	1	0	0	17	28	93
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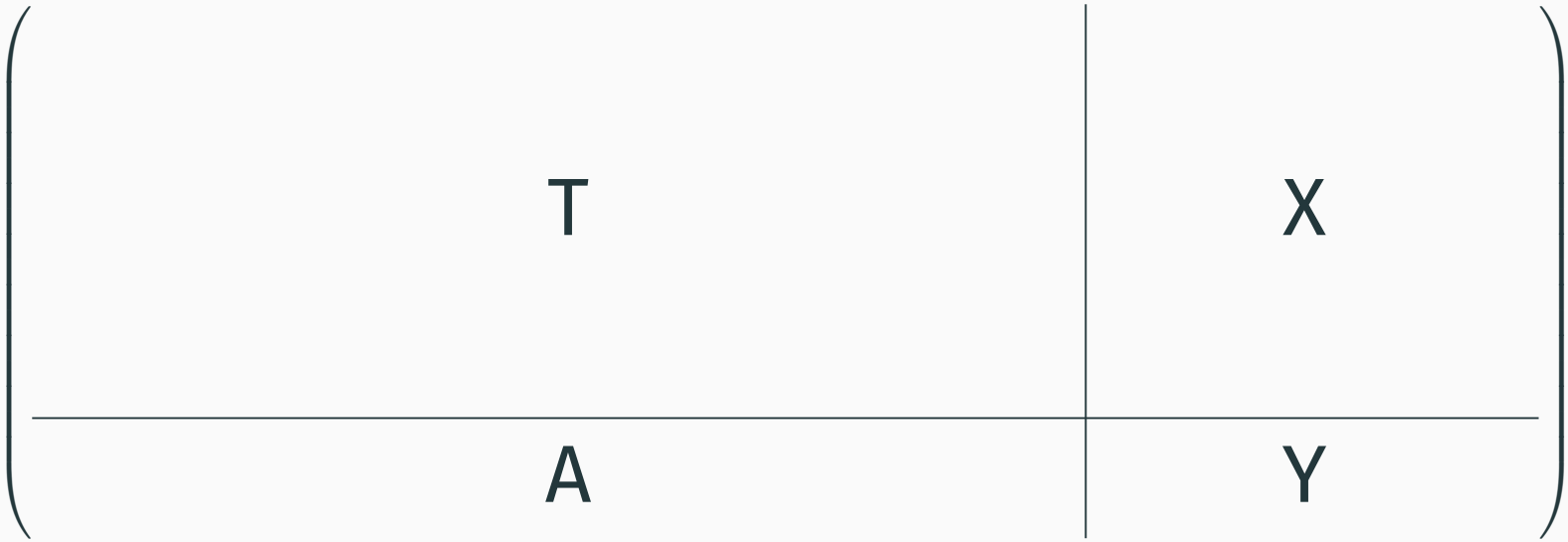
$x_1 \cdot f_3$	0	0	1	0	9	30	0	72	0	0

Upper bounds

$$\left(\begin{array}{ccccccc|ccc} 1 & 0 & 0 & 0 & 17 & 28 & 0 & 93 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 17 & 28 & 0 & 93 & 0 \\ 0 & 0 & 1 & 0 & 0 & 70 & 77 & 0 & 62 & 0 \\ 0 & 0 & 0 & 1 & 0 & 9 & 30 & 0 & 72 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 17 & 28 & 93 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 70 & 77 & 62 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 9 & 30 & 72 \\ \hline 0 & 1 & 0 & 0 & 70 & 77 & 0 & 62 & 0 & 0 \\ 0 & 0 & 1 & 0 & 9 & 30 & 0 & 72 & 0 & 0 \end{array}\right)$$

Upper bounds



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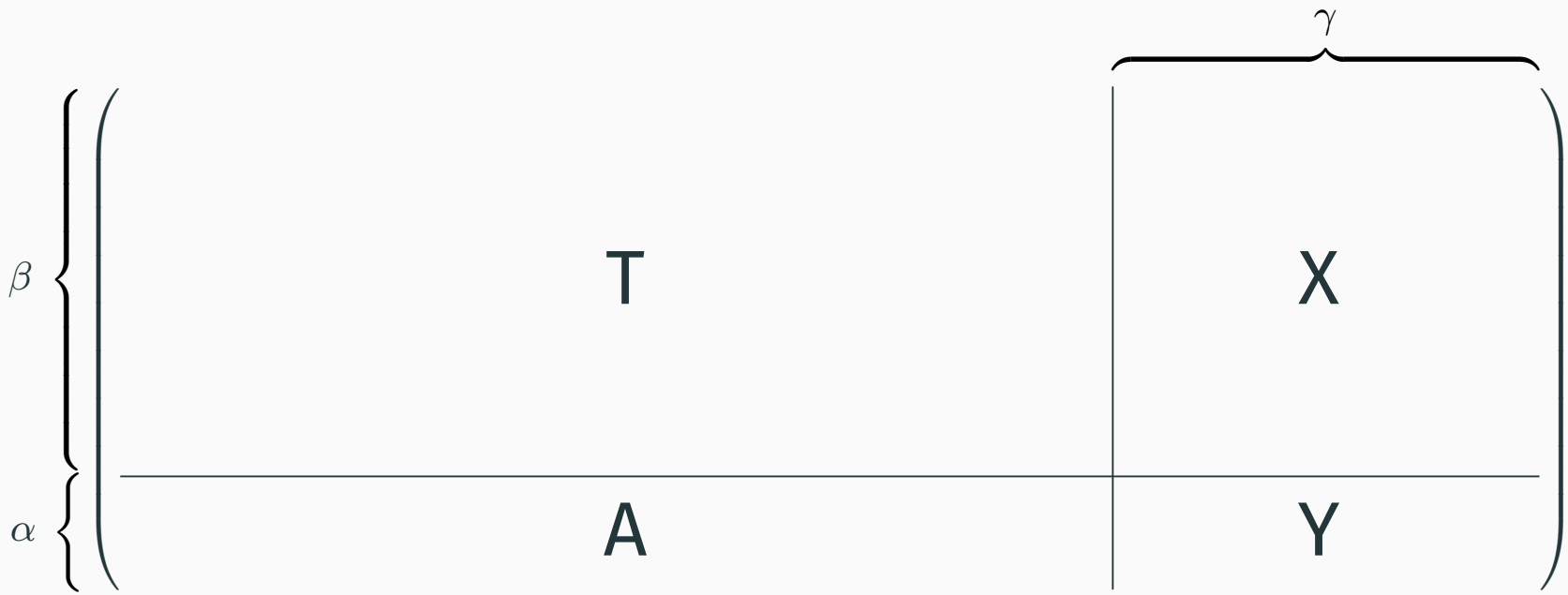

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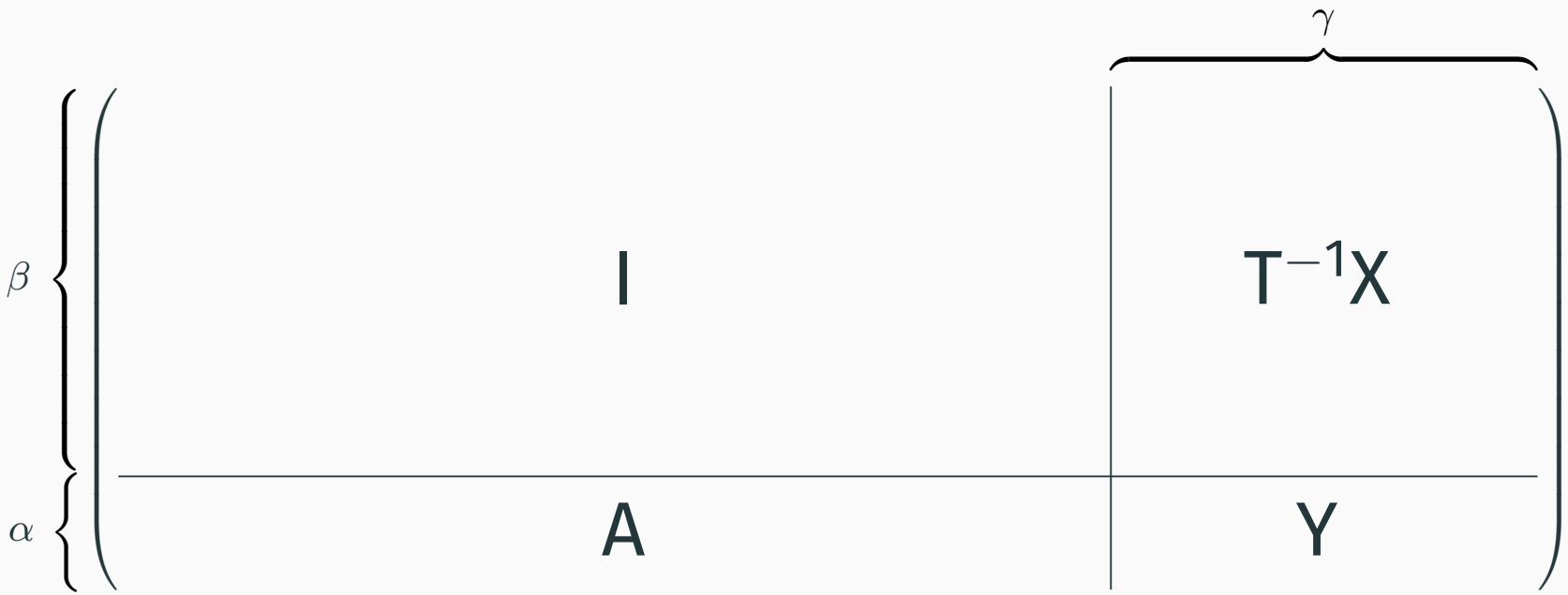

Step 1. Echelonize $(\mathsf{T} \mid \mathsf{X})$: $O(\beta^2\gamma^{\omega-2})$ [Jeannerod, Pernet, Storjohann, 2013]

Upper bounds

$$\begin{array}{c} \\ \beta \\ \alpha \end{array} \overbrace{\left(\begin{array}{c|c} I & T^{-1}X \\ \hline 0 & Y - AT^{-1}X \end{array}\right)}^{\gamma}$$

Step 1. Echelonize $(\mathsf{T} \mid \mathsf{X})$:	$O(\beta^2\gamma^{\omega-2})$	[Jeannerod, Pernet, Storjohann, 2013]
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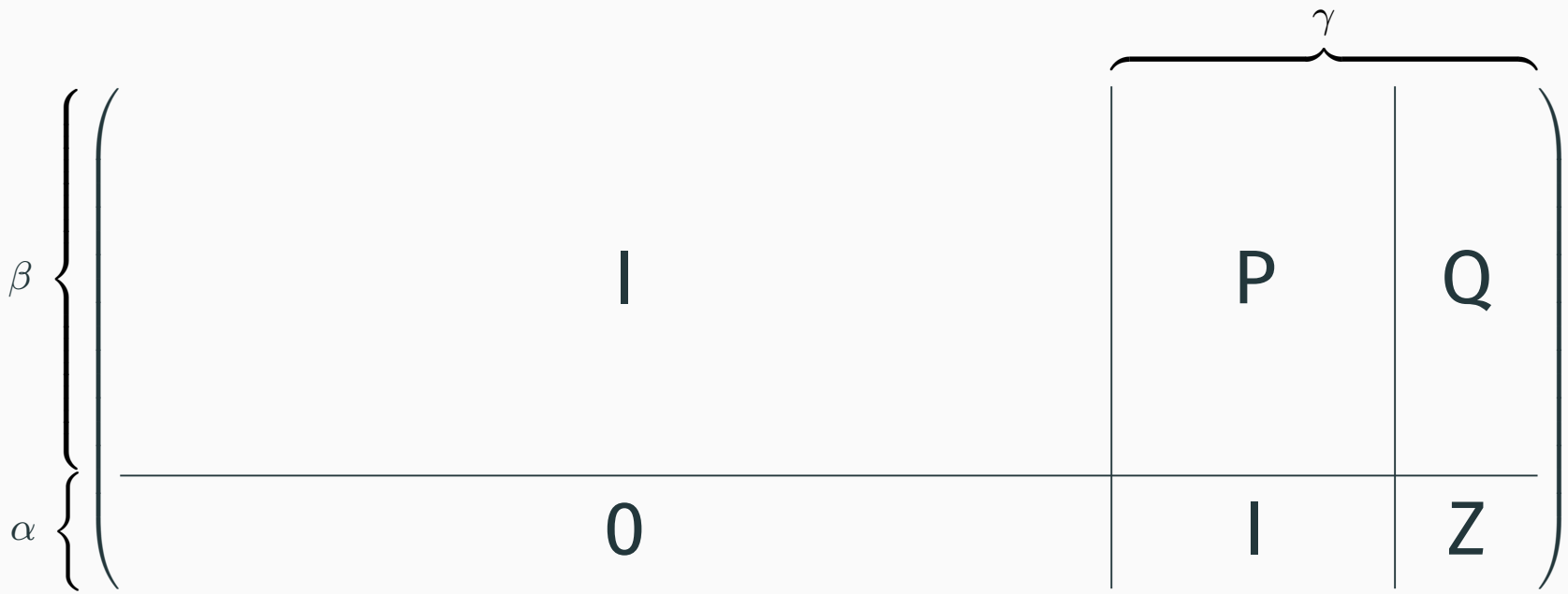
Upper bounds



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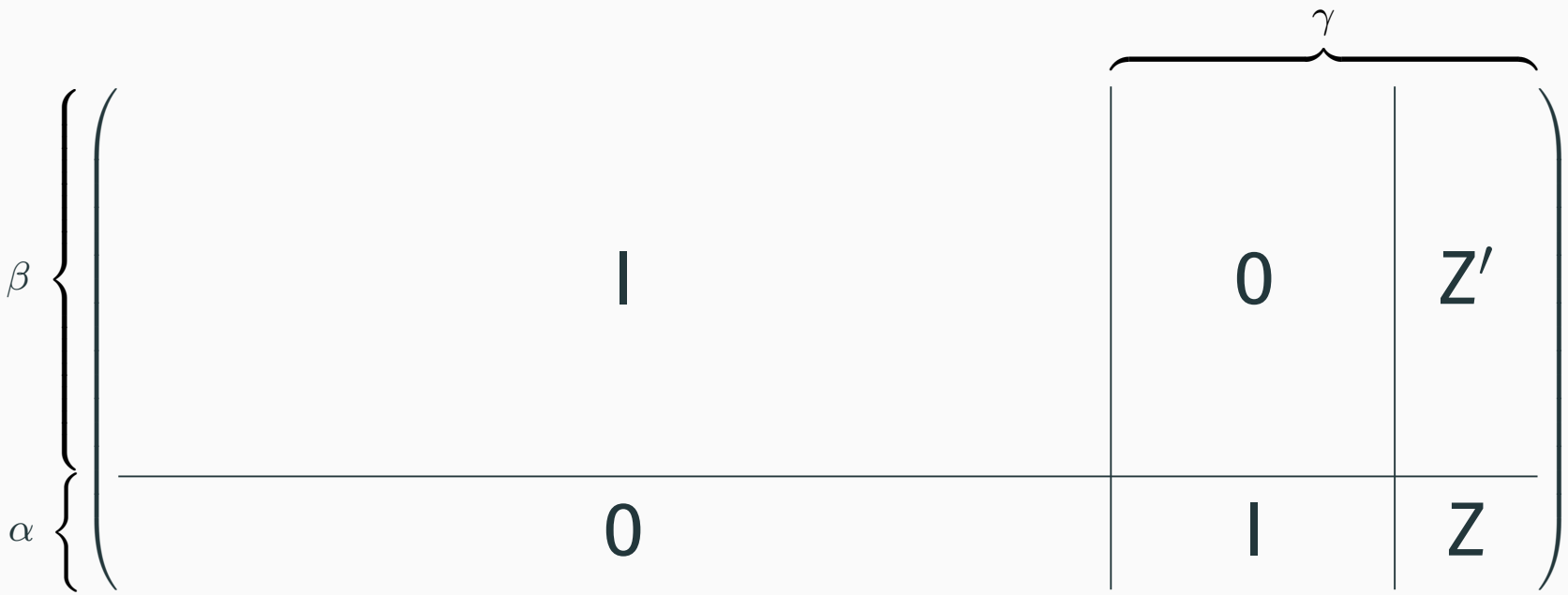

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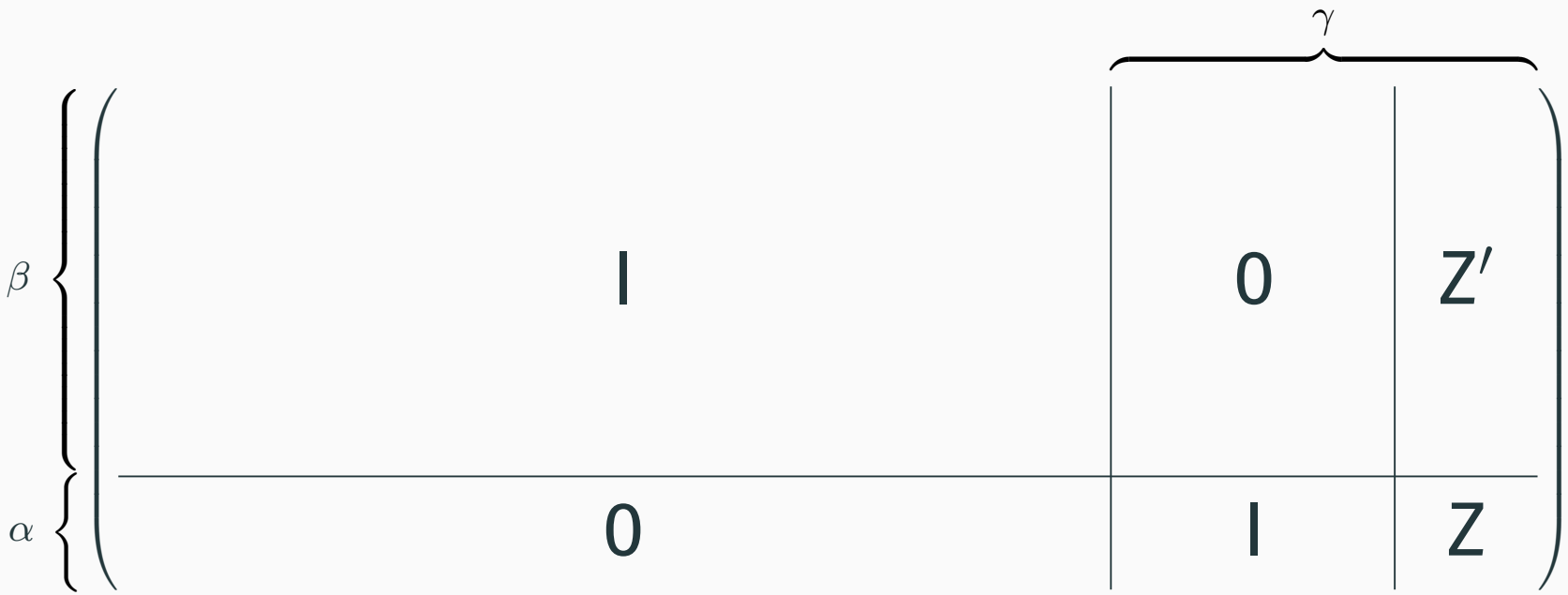

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Cost per degree: $O(\alpha^{\omega-2}\beta\gamma + \beta^2\gamma^{\omega-2})$

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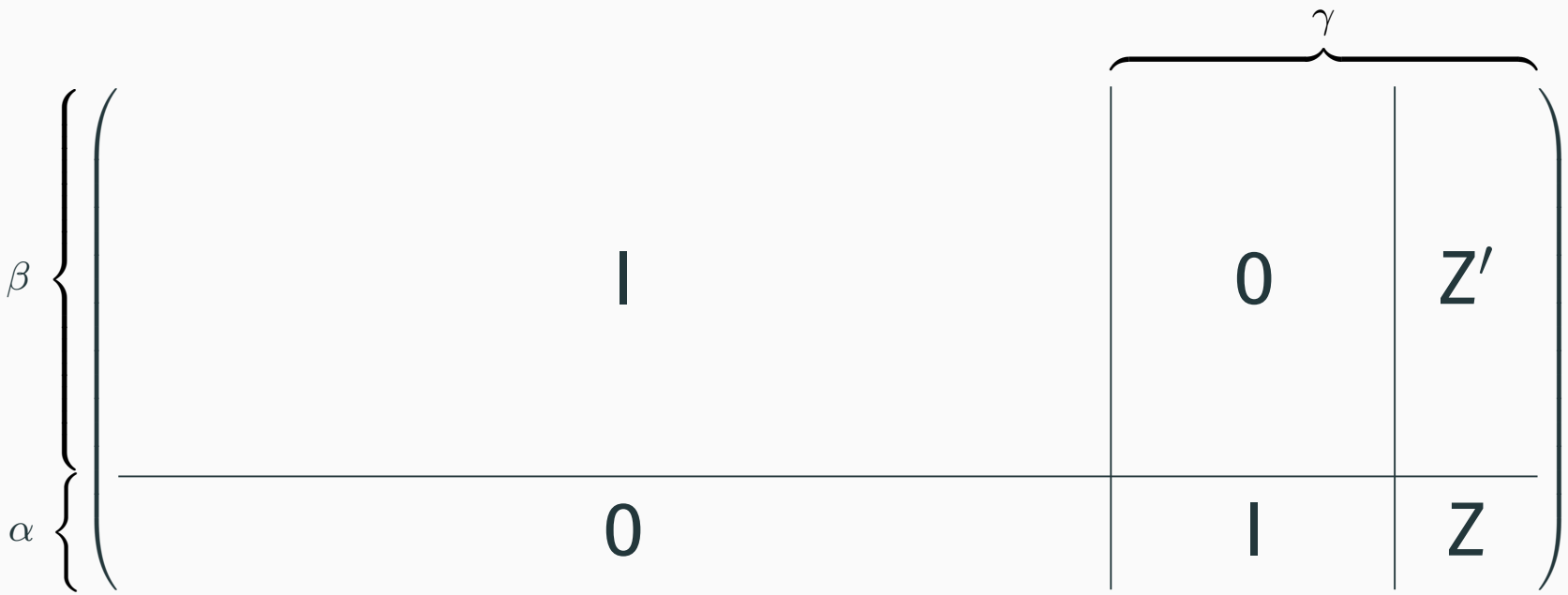

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Experimental results

n	r	k	D	d	rank	Std. F_5	Det. F_5
8	6	4	13	7	64	64	64
				8	130	256	130
				9	200	322	200
				10	276	385	276
				11	360	471	360
				12	454	559	454
				13	560	650	560
9	7	4	15	8	81	81	81
				9	164	324	164
				10	251	401	251
				11	344	486	344
				12	445	584	445
				13	556	675	556
				14	679	813	679
				15	816	931	816

n	r	k	D	d	rank	Std. F_5	Det. F_5
4	1	9	4	2	36	36	36
				3	164	324	164
				4	495	582	582
5	2	9	7	3	100	100	100
				4	450	900	450
				5	1278	1956	1956
				6	3002	3546	3546
				7	6435	6685	6685
6	3	9	6	4	225	225	225
				5	1017	2025	1017
				6	2838	4715	4715
7	4	9	6	5	441	441	441
				6	2009	3969	2009

n	r	k	D	d	rank	Std. F_5	Det. F_5
5	1	16	4	2	100	100	100
				3	800	1600	800
				4	3875	4662	4662
6	2	16	4	3	400	400	400
				4	3250	6400	3250

$k =$ number of variables.
$D =$ highest degree appearing in the (reduced) grevlex Gröbner basis for $F_r(M)$.

- When $r = n - 2$, all Macaulay matrices are full rank.
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$\sim 30\%$ of reductions to zero removed in general case

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- Efficient implementations of new algorithms.

Thanks. Questions?